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ACTA PHYSICA POLONICA

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M. Kunz

AFTER SIX YEARS OF WAR

After an interval of six years the Polish Physical Society resumes its activities aiming, as before the War, at the encouragement of physical research in this country. ACTA PHYSICA POLONICA, the Society's official organ, are called again to existence.

On the threshold of this new period we should realize what losses the science of physics had to suffer in Poland during the most terrible of all wars in which this country ever took part. In the first place we should consider death's heavy toll taken from among scientific workers, all the more regrettable, as in many cases they were victims of furious brutality of the enemy, who striving for the total destruction of the Polish nation persecuted our most eminent men with particular vehemence.

Here is the list of members of the Polish Physical Society and other Polish physicists deceased since the outbreak of the War.

Edward B e k i e r, professor of physical chemistry, Stephen Bathory University, Wilno, died in Wilno, 1945.

Mieczysław C e n t n e r s z w e r, professor of physical chemistry, first in Riga, later in Warsaw, a prominent representative of that branch of science, author of many scientific works and several handbooks, active member of the Polish Academy of Science and Letters; has been assassinated in Warsaw on July 27, 1944.

Jan C i c h o c k i, Ph. D., assistant at the university of Poznań, lately at the Institute of Experimental Physics of the University of Warsaw, had worked in Paris with Prof. J. P e r r i n and M. d e B r o g l i e; in the last years before the War he worked on problems of nuclear physics.

Bernard C z e m p l i k, Ph. D., lecturer in physics at the Polish Medical Department in Edinburgh, died in autumn 1944.

Ludwik D ą b r o w s k i, M. Phys., assistant at the University of Wilno, pupil and collaborator of the late Prof. J. P a t k o w s k i, published papers on band spectra, for a time worked at Katowice in the Silesian Department of the Polish Bureau of Standards.

Aleksander D m o c h o w s k i, an eminent teacher, director of the Physical Laboratory for Secondary Schools in Wilno, author of many excellent text-books, editor of the quaterly review «Physics and Chemistry at School».

Stanisław D o b i ń s k i, Ph. D., lecturer (docent) in experimental physics at the Jagellonian University in Cracow and the University of Poznań, had worked with Prof. Sir George T h o m s o n in London, published papers on different topics especially on diffraction of electrons and structure of metallic surfaces; as an officer of the Army Reserve, he died of his wounds received in the first days of the War in the defense of Warsaw.

Dobiesław D o b o r z y ń s k i, Ph. D., lecturer (docent) in experimental physics at the Jagellonian University in Cracow, worked on the polarization of dielectric media, later, after studies under Prof. W. H. K e e s o m at Leyden, also specialized in low temperatures; took part in the War as an officer of the Army Reserve, was arrested first in the famous man-trap arranged by the Germans against the Cracow professors in November 1939; in 1942 he was arrested once more, condemned to death, and executed.

Gustaw D o b o r z y ń s k i, Ph. D., teacher of mathematics and physics in Warsaw.

Jadwiga F a l k o w s k a, teacher of physics at the famous lyceum at Krzemieniec, formerly assistant at the Physical Institute of the Stephen Bathory University, Wilno.

Henryk H e r s z f i n k i e l, Ph. D., callaborator of the late Prof. L. W e r t e n s t e i n in Warsaw, worked on radioactivity.

Lejb Dawid H e r s z m a n, M. Phys., assistant to Prof. H. N i e w o d n i c z a ń s k i at the Stephen Bathory University at Wilno, commited suicide in the German established Ghetto in Wilno, 1941.

Józef H r y n k i e w i c z, M. Phys., teacher of physics in the Sigismund August Lyceum in Wilno, formerly assistant at the Physical Institute of the Stephen Bathory University, Wilno, died 1944.

Stanisław K a l a n d y k, professor of experimental physics at the Faculty of Medicine of the Poznań University, had worked in the Cavendish Laboratory under J. J. T h o m s o n; most of his publication concern the properties of ions and electrons in gases as well as divers phenomena associated with explosions; has been assasinated by the Germans at Poznań in the first days of the War.

Antoni K a r p o w i c z, Ph. D., formerly assistant to Prof.. S. K a l i n o w s k i at the High Technical School, Warsaw.

Leon K l e c k i, teacher of physics at the french lyceum in Warsaw, a pupil of K u n d t; among other works he also wrote an essay on the late Prof. L. N a t a n s o n.

Kamil K r a f t, Ph. D., M. D., physicist and physician, published papers on relativity and optics, his paper on an interference-colour scale is often quoted. Died in Cracow, 1945.

Juda K r e i s l e r, Ph. D., assistant to Prof. W. R u b i n o w i c z at the John Casimir University, Lwów.

Stefan K r e u t z, professor of mineralogy at the Jagellonian University, Cracow, active member of the Polish Academy of Science and Letters, died in 1941.

Tadeusz K u c z y ń s k i, professor of physical chemistry at the High Technical School, Lwów, murdered by the Germans in 1941.

Hilary L a c h s, professor of physical chemistry at the Free University, Warsaw.

Andrzej Ł a s t o w i e c k i, Ph. D., assistant do Prof. S. L o r i a at the John Casimir University, Lwów.

Myron M a t h i s s o n, Ph. D., lecturer (docent) in natural philosophy at the University of Warsaw, author of works on relativity and quantum mechanics, died in Cambridge, 1940.

Tadeusz M o d z e l e w s k i, electrical engineer, assassinated by the Germans during the Warsaw Insurection, 1944.

Tadeusz N a y d e r, Ph. D., teacher of physics at the School of Industry, Cracow, formerly assistant at the Physical Laboratory of the Jagellonian University, Cracow, author of some experimental works, died in Cracow, 1945.

Józef P a t k o w s k i, professor of experimental physics at the Stephen Bathory University, Wilno, author of works on radioactivity as well as on band spectra (isotope effect); the latter were partly executed at Newcastle in collaboration with Prof. W. E. C u r t i s s; the last years before the War he filled the post of director of the Department of Science, Letters and Academic Schools in the Ministry of Education; fell a victim of an air-bombardment of Warsaw, September, 1943.

Stanisław P i l a t, professor of technology of fluid fuels at the High Technical School, Lwów, murdered by the Germans in 1941.

Mieczysław P o ż a r y s k i, professor of electrical technology at the High Technical School, Warsaw.

Antoni P r z e b o r s k i, Ph. D., published some experimental works on molecular spectra.

Antoni R a a b e, M. Phys., collaborated with Prof. J. W e y s s e n h o f f during the War. Captured in a man-hunt in the streets of Cracow, died in the concentration camp of Oświęcim, 1942.

Irena R a m m, *née* M a n t e u f f e l, M. Phys., for a time assistant at the Institute of Experimental Physics of the Warsaw University, worked on X-rays analysis in Warsaw, later in Paris under Mme C u r i e on radioactivity.

Zofia R o t s z a j n, Ph. D., formerly teacher of physics in secondary schools in Warsaw.

Aleksander S i k o r a, M. Phys., pupil and assistant of Prof. S. P i e ń k o w s k i, University of Warsaw, killed at Katyń.

Ireneusz Ś l u s a r c z y k, M. Phys., pupil and for a time assistant of Prof. S. P i e ń k o w s k i, University of Warsaw, worked afterwards in the State Meteorological Institute and published papers on meteorology; died in the concentration camp of Majdanek.

Zdzisław S p e c h t, Phil. D., assistant to Prof. S. L o r i a at the John Casimir University, Lwów, died in 1943.

Oskar S t e l m a n, assistant to Prof. M. G r o t o w s k i at the Free University, Warsaw.

Leon S t ę p i ń s k i, teacher of physics and chemistry in secondary schools, Wilno.

Karol S z l e n k e r, Ph. D., pupil of W. R o e n t g e n, specialized in optics, assassinated by the Germans during the Warsaw Insurection, 1944.

Witold T r y l s k i, M. Phys., hanged by the Germans in Warsaw in 1943 as one of fifty hostages.

Tadeusz T u c h o l s k i, Ph. D., lecturer (docent) in chemistry of explosives at the High Technical School, Warsaw, pupil and collaborator of the late Prof. S. K a l a n d y k, University of Poznań; 1934/35 worked with Prof. R i d e a l in the Department of Colloid Science in Cambridge on deuterium; took part in the campaign of 1939 as officer of the Army Reserve, killed at Katyń.

Ludwik W e r t e n s t e i n, professor of physics at the Free University in Warsaw and Łódź, director of the Radiological Laboratory of the Warsaw Scientific Society, pupil of Mme C u r i e, also worked with Lord R u t h e r f o r d in Cambridge, published many scientific works mainly on natural and artificial radioactivity; a prominent teacher, he was also a gifted writer of popular books on scientific topics; having translated into Polish Mme C u r i e's «Radioactivity» he supplemented it with a valuable appendix written by himself. Killed on January 18, 1945 by a shell splinter in Budapest, where he had taken refuge from German persecutions in Poland.

Bruno W i n a w e r, Ph. D., pupil of P. L e n a r d, sometime assistant to Prof. J. K o w a l s k i, University of Warsaw, distinguished Polish comedian, whose plays have been translated into many languages, he was also well known as speaker by radio and writer of popular articles on scientific topics.

Mojsiej Żyw, Ph. D., collaborator of the late Prof. L. Wertenstein. Published works on radioactivity; together with Danysz he discovered Radiofluorine ($^{17}_{9}F$), perished at the slaughter of the Warsaw Ghetto, 1943.

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The above brief mentions cannot give but an inadequate picture of the importance of our losses. The works of many members of our Society whose names appears on this list deserve fuller appreciation. Their merits will be better considered in commemorative notes to appear later.

Besides personal losses also the damage to scientific equipment is enormous. Of all Polish physical institutions which were working before the War the following are deemed to exist, though as a matter of fact their existence in many cases is more nominal than real. (1) Warsaw: The Institute of Experimental Physics of the University, the Experimental Laboratory under the supervision of the professor of natural philosophy, two physical institutes of the High Technical School, the Radiological Laboratory of the Warsaw Scientific Society, and the Geophysical Observatory at Świder. (2) Cracow: The Physical Institute of the Jagellonian University and the Physical Institute of the Mining Academy. The Physical Institute of the Jagellonian University — founded at the end of the XVIII-th century — is the most ancient physical institution in Poland and has been rendered famous in the past by men like Zygmunt Wróblewski, August Witkowski and Marian Smoluchowski. (3) Poznań: Three physical institutes, of which two belong to the Faculty of Science, the third to the Faculty of Medicine of the University of Poznań.

The best equipped of the above was the excellently appointed Institute of Experimental Physics of the Warsaw University, from which a great number of works — chiefly concerning molecular optics — appeared during the twenty years preceding the War. In the field of photoluminescence the Institute enjoyed world-wide renown. Now, it has practically ceased to exist. Only the building remains, but it requires important renovation. All the precious apparatus and instruments have been robbed by the Germans.

The laboratory associated with the chair of natural philosophy has been completely burnt down. Other Warsaw laboratories as well as both institutions in Cracow have also been badly devastated; nothing or scarcely anything of their pre-war belongings has remained.

The Institute of the Jagellonian University in Cracow succeded in saving its valuable collection of books.

It is clear that in the present position all activities shall have to concentrate on teaching and organization, whereas research work shall have for a time to be restrained from almost completely. One must also reckon with the fact that not only purely scientific institutions have been heavily damaged, but also the manufacture of products indispensable for experimental investigation has badly suffered. The chemical, optical, and electrical industries as well as ordinary mechanical workshops are in a state of utter devastation, and besides they cannot get supplies of the most important materials.

All those difficulties and hindrances shall not discourage us. We shall spare no effort towards the restitution of conditions required for the normal development of physics in this contry. Science in Poland had already passed through many a difficult crisis and every time succeeded to emerge to a new and more successful existence. We firmly believe that now again we shall manage to attain our goal, if we only get due help (which was many times promised) either in the form of indemnities or in friendly loans or gifts from abroad. A certain quantity of books has been already gladly received. We hope that the present will contribute to the conviction that the help will not be wasted.

Cracow, May 1945. *Konstanty Zakrzewski*

We regret to announce the following deaths of members of the Polish Physical Society since May 1945:

Mr. Leon Ćwikliński, M. Phys., assistant at the physical laboratory of the Mining Academy, Cracow, on January 2, 1946.

Prof. Tadeusz Pęczalski, professor of natural philosophy in the University of Poznań, died in Paris on February 2, 1947.

Prof. Stanisław Kalinowski, professor of physics at the High Technical School, Warsaw, director of the Geophysical Observatory at Świder, honorary member of the Warsaw Museum of Industry and Agriculture, on March 27, 1946.

Dr. Józef Lubański, assistant in the aerodynamical laboratory of Prof. J. M. Burgers at Utrecht, on December 11, 1946.

Mrs. Maria Mączyńska, *née* Moraczewska, M. Phys., sometime assistant at the Institute of Experimental Physics of the University of Warsaw, on December 24, 1946.

RELATIVISTIC DYNAMICS OF SPIN-FLUIDS AND SPIN-PARTICLES*

By Jan WEYSSENHOFF and A. RAABE †, Institute of Theoretical Physics, Jagellonian University, Kraków **

1. Following up a train of thought inaugurated by Einstein and Gromer, Mathisson *(1, 1937)* and Lubański *(2, 1937)* deduced the equation of motion of a material particle endowed with spin from the general principles of the theory of relativity. Even for a free particle in Galileian domains these equations do not coincide with the Newtonian laws of motion; there remains an additional term depending on the internal angular momentum or spin of the particle, which raises the order of these differential equations to three. Mathisson did not notice the fact that for the above conditions his equations are equivalent to the equations which were previously found by Frenkel *(3, 1926)* for a spinning electron (if only their terms depending on the electromagnetic field are dropped). This is all the more excusable as Frenkel considered throughout the additional term in his equations as an infinitely small perturbation, and did not even mention the fact that they disagree with Newton's First Law for a free particle.

In the present paper we give a third method of obtaining the same equations by establishing first the laws of the dynamics of an incoherent spin-fluid and passing then to the limit. Strictly speaking, we obtain different and much simpler equations, which prove however to be equivalent to Frenkel's and Mathisson's. The simplification is due to the explicit introduction of the 4-vector of linear monumentum and energy G^{α}.

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* Presented at a meeting of the Cracow Section of the Polish Physical Society on February 28, 1945. The main contents of this paper and the next one, as well as most of the results of the three following, were a subject of a lecture at a secret meeting of physicists at Prof. Pieńkowski's home in Warsaw, October 1942.

** Mr. Raabe was a highly gifted young physicist with whom I outlined in all its main features the contents of this paper and the next one in 1940/41 in Lwów. We tried to pursue our work in 1942 in Cracow, but unfortunately in June 1942 Mr. Raabe fell victim of a man-hunt in the streets of Cracow; he died four months later in the German concentration camp of Oświęcim. (J. W.)

In Sections 2—4 we develop the dynamics of a spin-fluid by a method similar to that usually applied in the reliativistic dynamics of incoherent matter (without internal angular momentum), the main difference consisting in the assumed asymmetry of the energy tensor $T^{\alpha\beta}$. In Section 5 we obtain the Frenkel-Mathisson equations by integrating the equations of motion of a spin-fluid over an infinitely small element of volume. In Section 6 we show that the most general motion of a free spin-particle in Galileian domains, where the special theory of relativity prevails, is a relativistic superposition of a translation and a uniform circular motion. M a t h i s s o n proved the same result for the non-relastivistic case only. Finally, in Section 7 we write down the equations of motion of a spin-particle with an electric charge and a magnetic moment moving in an electromagnetic field.

2. We base the dynamics of a spin-fluid in the special theory of relativity on two fundamental principles: the principle of conservation of energy and linear monunumentum, and the principle of conservation of angular momentum. By spin-fluid we mean a fluid each element of which possesses besides energy and linear momentum also a certain amount of angular momentum, proportional — just as the energy and the linear momentum — to the volume of the element.

We represent the density of angular momentum per unit rest-volume[1] by the four-dimensional bivector (antisymmetric tensor of rank two) $s^{\alpha\beta}$. Its three space components form a three-dimensional vector

$$\hat{s} = \{s^{23},\ s^{31},\ s^{12}\} \tag{1}$$

equal in the coordinate system in which the fluid momentary rests to the three-dimensional density of angular momentum. Its three space-time components form also a three-dimensional vector, which we shall denote by

$$\hat{q} = \{s^{14},\ s^{24},\ s^{34}\}. \tag{1'}$$

We shall assume that this vecor vanishes in the rest-system Σ_0 of the fluid. The four-dimensional tensor expression of that condition is[2]

$$s^{\alpha\beta} u_\beta = o, \tag{2}$$

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[1] The element of rest-volume dV_0 is defined as the element of the three-dimensional orthogonal cross-section of a world-tube. Multiplied by (c and) the element of proper time along this world-tube, τ, it yields the four-dimensional element of volume $d\Omega = dx^1dx^2dx^3dx^4 = dV dx^4 = c\,dV dt = c\,dV_0\,d\tau$. Hence, $u^4 = dt/d\tau = 1/\sqrt{1-\beta^2} = dV_0/dV$, where $\beta = v/c$.

[2] With regard to the notations used, we observe what follows. Greek indices take the values 1 to 4, latin 1 to 3. Zero is never used, as a tensorial index, as

as $u_k = o$ in Σ_0, and of the fourfold sum there remains only the term $s^{i4} u_4$, the vanishing of which yields $s^{i4} = q^i = o$.

The vectors $\hat{s}$ and $\hat{q}$ transform in the same manner as the magnetic and electric field intensities, or the magnetic and electric dipole moments. The condition (2) for a magneto-electric moment amounts to the condition that this moment should be «purely magnetic», i. e., that its electric component should vanish in the coordinate-system in which the element of volume is momentary at rest.

We may also write equation (2) in three-dimensional vector form, as follows:

$$\hat{q} = \frac{1}{c} v \times \hat{s}, \tag{2'}$$

the fourth equation $\hat{q} \cdot \hat{v} = o$ being a consequence thereof.

We express now the law of conservation of energy and momentum in the familiar form

$$\partial_\beta T^{\alpha\beta} = o, \tag{3}$$

where $T^{\alpha\beta}$ is the momentum-energy tensor. However, in contradistinction to what has been done hitherto, we do not assume the symmetry of the tensor $T^{\alpha\beta}$, as this symmetry — as will be presently shown — is equivalent to the vanishing of the intrinsic angular momentum $s^{\alpha\beta}$. Instead of writing $T^{\alpha\beta} = \mu_0 u^\alpha u^\beta$ as usual for incoherent matter, we put more generally

$$T^{\alpha\beta} = g^\alpha u^\beta, \tag{4}$$

where g^α is the four-vector of the proper density of linear momentum

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it is reserved for special uses, e. g. for labelling quantities in the rest system (or else for initial values, etc.). Letters with a circumflex accent are used to denote three-dimensional vectors (the circumflex accent is less used than the horizontal arrow, but it is more convenient in print as well as in writing; it may be conceived as an arrow pointing upwards). ∂_α is short for $\partial/\partial x^\alpha$. The signature of the Minkowski space is taken to be $+++-$, i. e. $g_{11} = g_{22} = g_{33} = 1$, $g_{44} = -1$, $g_{\alpha\beta} = 0$ ($\alpha \neq \beta$); consequently raising and lowering of the index 4 changes the sign. $x^4 = ct$, $ds^2 = -c^2 d\tau^2$, $d\tau$ is the element of the proper time along the world-line of matter; $u^\alpha = dx^\alpha/d\tau$ is the four-dimensional velocity. — As far as possible we follow the general rule that all the components of any four-dimensional tensor should have the same dimension, this dimension being also the same as that of the three-dimensional quantity after which the tensor has been called. Thus all the four-dimensional formulae become as similar as possible to the corresponding three-dimensional ones, and wherever c has been put equal to 1 to make calculations easier, it is quite simple to restore it on dimensional homogeneity grounds. See also reference (7).

and energy.[3] Of course, we do not postulate *a priori* that g^α is parallel to u^α.

Taking into account the relation between the components of g^α and of the three-dimensional density of linear momentum $\hat{g} = \{g^1, g^2, g^3\}$, we may write

$$T^{\alpha\beta} = \left\| \begin{array}{c:c} g^i v^k & c\hat{g} \\ \hdashline c\mu\hat{v} & c^2\mu \end{array} \right\| \tag{5}$$

Inserting (5) in (3), we get the familar equations expressing the conservation of linear momentum and energy in the relativistic dynamics of continuous media.

It seems worth mentioning that (a) we assume (3) but not $\partial_\alpha T^{\alpha\beta} = 0$, a relation which is identical with (3) in the case of a symmetric $T^{\alpha\beta}$, and (b), by interpreting $\hat{v}$ in (5) as representing the velocity of the fluid, we assume that the energy does not flow in the rest-system of the fluid.

3. Before proceeding further, we shall put the equations (3) and (4) together into a new form. In the non-relativistic dynamics of continuous media two different differentiations with respect to the time are used in addition to the «local differentiation», i. e. the differentiation with respect to the time at constant x, y, z, which we shall denote by ∂_t. These two additional differentiations are defined as follows:

$$d_t f = \partial_t f + v^k \partial_k f, \qquad f = f(x, y, z), \tag{6}$$

$$D_t f = d_t f + f \partial_k v^k = \partial_t f + \partial_k (f v^k). \tag{7}$$

The first, which is in more common use, is the derivative at constant Langrage coordinates; sometimes it is called «substantial derivative» or time derivative «following the particle». The second might be called «time derivative for densities»; it possesses the following caracteristic property

$$d_t (f\, dV) = (D_t f)\, dV, \tag{8}$$

that is

$$d_t \int f\, dV = \int (D_t f)\, dV. \tag{8'}$$

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[3] The three-dimensional density of linear momentum $\hat{g} = \{T^{k4}\}/c$ is defined as the momentum per unit volume in the coordinate system of interest, whereas g^α ($= \mu_0 u^\alpha$ in spinless dynamics) is taken per unit of rest volume. Therefore $g^\alpha = \sqrt{1-\beta^2}\,\{\hat{g}\,,\, w/c\}$, where $w = \mu c^2$ is the density of energy and μ the density of matter. As the four-dimensional velocity $u^\alpha = \{\hat{v} \mid c\}/\sqrt{1-\beta^2}$, it follows that $g^i u^k = g^i v^k$. The well known formula $\mu_0 = \mu(1-\beta^2)$, which is an immediate consequence of the relation $g_\alpha u^\alpha = inv.$, gives the relation between μ and the proper density of matter μ_0.

We now define two analogous operators in the Minkowski space M_4:

$$d_\tau f = \dot{f} = u^\nu \partial_\nu f, \quad f = (x^1, x^2\, x^3, x^4), \tag{9}$$

$$D_\tau f = d_\tau f + f \partial_\nu u^\nu = \partial_\nu (f u^\nu). \tag{10}$$

The first one reduces to the usual derivative with respect to the proper time τ if we define a new function $f(\tau)$ taking the same values as $f(x^\alpha)$ on the world-line of the particle; we shall denote it also by a dot over the letter. The derivative defined in (10) appears in the following four-dimensional formula,[4] analogous to the three-dimensional formula (8):

$$d_\tau (f\, d\Omega) = (D_\tau f)\, d\Omega, \tag{11}$$

where $d\Omega$ is the four-dimensoinal element of volume $dx^1\, dx^2\, dx^3\, dx^4$. By expressing $d\Omega$ as a product of the element of proper time $d\tau$ and the element of proper volume dV_0 we get from (11) a new formula, which we shall need in section 5,

$$d_\tau (f\, dV_0) = (D_\tau f)\, dV_0. \tag{12}$$

It can be verified without trouble that

$$D_t g^i = D_\tau g^i. \tag{13}$$

We are now in position to write down the equations (3) in the new form. As

$$\partial_\beta T^{\alpha\beta} = \partial_\beta (g^\alpha u^\beta) = D_\tau g^\alpha, \tag{14}$$

we may write instead of (3)

$$\boxed{D_\tau g^\alpha = o.} \tag{15}$$

This form is well suited to immediate integration (over a sufficiently small volume of the spin-fluid).

4. In the relativistic dynamics without internal spin there exist besides the 4 fundamental equations (3) 6 relations $T^{\alpha\beta} = T^{\beta\alpha}$, which

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[4] In analogy with the three-dimensional case this formula may be deduced by transforming the proper-time derivative of the Jacobian $\dfrac{\partial\, (x^1, x^2, x^3, x^4)}{\partial\, (\xi^1, \xi^2, \xi^3, \xi^4)}$, where ξ^1, ξ^2, ξ^3 are the Lagrange coordinates characterizing the individual world lines of matter, and ξ^4 is an arbitrary parameter along those lines. The situation in the Minkowski four-dimensional space is analogous to the case of stationary flow in Euclidean space. To emphasize this analogy still further one has to write $x^\alpha = x^\alpha (\xi^1, \xi^2, \xi^3, \xi^4 + \tau)$, $u^\alpha = \partial x^\alpha / \partial \tau$ $(\xi^\alpha = \text{const})$, but the argument may be also carried through with the notations $x^\alpha = x^\alpha$ $(\xi^1, \xi^2, \xi^3, \tau)$, $u^\alpha = \partial x^\alpha / \partial \tau$ $(\xi^k = \text{const})$.

express the symmetry of the energy tensor $T^{\alpha\beta}$. In the case of lack of that symmetry other equations must appear to fix the antisymmetrical components of $T^{\alpha\beta}$. This role is taken over by the generalized law of conservation of angular momentum, which may be obviously stated as follows:

$$D_\tau (x^\alpha g^\beta - x^\beta g^\alpha) + D_\tau s^{\alpha\beta} = o. \tag{16}$$

The first term refers to the density of external angular momentum, i. e. to the moment of linear momentum density; the second to the density of internal angular momentum or spin. Now, applying the following general rule (which is an immediate consequence of the equations of definition (9) and (10)):

$$D_\tau (fg) = f D_\tau g + g d_\tau f = f d_\tau g + g D_\tau f, \tag{17}$$

and taking into account (15), we may write instead of (16)

$$\boxed{D_\tau s^{\alpha\beta} = T^{\alpha\beta} - T^{\beta\alpha} = g^\alpha u^\beta - g^\beta u^\alpha.} \tag{18}$$

Thus, the existence of the internal angular momentum is connected with the asymmetry of the energy tensor. It is also connected with the existence of a transversal linear momentum, i. e., of a component of the 4-vector of linear momentum density g^α perpendicular (in M_4) to the four-dimensional velocity u^α this last result being a consequence of what follows. Multiplying (18) by u_β, bearing in mind that $u_\beta u^\beta = - c^2$, and putting

$$\mu_0 = - \frac{1}{c^2} u_\beta g^\beta, \tag{19}$$

we get the following relation between the 4-vectors of linear momentum and energy g^α, velocity u^α, and acceleration $\dot{u}^\alpha$:

$$\boxed{g^\alpha = \mu_0 u^\alpha - u_\beta D_\tau s^{\alpha\beta} = \mu_0 u^\alpha + s^{\alpha\beta} \dot{u}_\beta.} \tag{20}$$

The second equality may be easily demonstrated by taking into account the result of applying the operator D_τ to equation (2) according to rule (17).

5. The equations of motion of a material particle with spin may be now obtained by integrating the equations (3), (15), (18) and (20) over a volume so small that we may consider in it u^α and $\dot{u}^\alpha$ as constants.

Putting

$$G^{\alpha} = \int g^{\alpha} dV_0, \tag{21}$$

$$S^{\alpha\beta} = \int s^{\alpha\beta} dV_0, \tag{22}$$

and

$$m_0 = \int \mu_0 dV_0, \tag{23}$$

we get thus

$$S^{\alpha\beta} u_\beta = 0, \tag{24}$$

$$\dot{G}^{\alpha} = 0, \tag{25}$$

$$\dot{S}^{\alpha\beta} = G^{\alpha} u^{\beta} - G^{\beta} u^{\alpha}, \tag{26}$$

$$G^{\alpha} = m_0 u^{\alpha} + \frac{1}{c^2} S^{\alpha\beta} \dot{u}_\beta, \tag{27}$$

where

$$m_0 = -\frac{1}{c^2} u_\beta G^\beta. \tag{28}$$

As before, equation (27) follows from (26) by multiplying it by u_β and taking into account (28) and the once differentiated relation (24).

Thanks to (27), we may eliminate G from the two preceding equations, and write

$$m_0 \dot{u}^{\alpha} + \frac{1}{c^2} S^{\alpha\sigma} \ddot{u}_\sigma = 0, \tag{29}$$

$$\dot{S}^{\alpha\beta} = \frac{1}{c^2} S^{\alpha\sigma} \dot{u}_\sigma u^\beta - \frac{1}{c^2} S^{\beta\sigma} \dot{u}_\sigma u^\alpha. \tag{30}$$

In (29) the term $\dot{S}^{\alpha\sigma} \dot{u}_\sigma$ has been dropped as it may be readily proved to vanish, by multiplying (26) by $\dot{u}_\beta$ and taking into account that $u_\beta \dot{u}^\beta = 0$ and $G_\beta \dot{u}^\beta = 0$ as a consequence of (27).

The equations (29) and (30) are Mathisson's equations referred to above in the introduction. They coincide also with Frenkel's equations of motion of a spinning electron (without external electromagnetic field),[5] when we bear in mind that his auxiliary vector a^α is equal to $-\frac{1}{\varkappa c^2} \dot{u}^\alpha$, where $\varkappa$ is the ratio of the magneto-electric to the mechanical moment. The simplification resulting from the explicit introduction of G^α is obvious.

Differentiating equation (28) with respect to the proper time τ, and paying due regard to the equations (25) and (27), we get

$$\dot{m}_0 = 0. \tag{31}$$

[5] J. Frenkel, *(2. 1926)*, equations (21) and (13a).

Hence, m_0 is a constant, and it may be rightly called «rest-mass of the spin-particle».

Equation (26) multiplied by $S_{\alpha\beta}$ yields, owing to (24), $\dot{S}_{\alpha\beta} S^{\alpha\beta} = 0$, and hence

$$S_{\alpha\beta} S^{\alpha\beta} = \hat{S} \cdot \hat{S} - \hat{Q} \cdot \hat{Q} = S_0^2 = \text{const.} \tag{32}$$

Here $\hat{S} = \{S^{23}, S^{31}, S^{12}\}$ is the three-dimensional vector of internal angular momentum of the particle and $\hat{Q} = \{S^{14}, S^{24}, S^{34}\}$. Hence, the magnitude of the internal angular momentum in the restsystem of the particle is constant.

By (25), G^α of a free particle is constant, and the same applies to M_c which we introduce by the following relation

$$G_\alpha G^\alpha = -M_c^2 c^2. \tag{33}$$

When G^α is a time-like vector, M_c is a real constant (which we may assume to be positive).

6. M a t h i s s o n *(4, 1937)* has succeded in integrating the equations of motion of a free particle only in the «non-relativistic» case, when v — but not a, the acceleration — is treated as an infinitesimal quantity. We are now going to show that the equations (24)—(27) may be exactly integrated in the general case, the only assumption needed being the time-like character of the four-vector G^α. We may then find an inertial frame of reference, Σ_c, in which $\hat{G}$ vanishes; we shall call it «proper system of the circle». In that system (for $c = 1$)

$$\hat{G} = \{G^1, G^2, G^3\} = 0. \quad G^4 = Mc, \tag{34}$$

(26) and (32) yield

$$\dot{S}^{ik} = 0, \quad \hat{S} = \text{const}, \tag{35}$$

and (28) yields

$$m_0 = M_c u^4 = \frac{M_c}{\sqrt{1 - v^2}}. \tag{36}$$

It follows that v, the scalar velocity of the particle, and u^4, the fourth component of its four-velocity, are constant in Σ_c, and hence

$$\dot{u}^4 = 0. \tag{37}$$

Thus not only the three first components of u^α are proportional to the components of $\hat{v}$, but also the space components of the four-acceleration $\dot{u}^\alpha$ are proportional to the components of the three-dimensional acceleration $\hat{a}$:

$$v^k = \frac{dx^k}{dt} = \frac{u^k}{u^4}, \quad a^k = \frac{d^2 x^k}{dt^2} = \frac{\dot{u}^k}{(u^4)^2}. \tag{38}$$

Equations (27), (32) and (35) yield now

$$m_0 u^k + S^{kl} \dot{u}_l = o.$$

Transforming the derivatives with respect to the proper time into the derivatives with respect to t, taking into account (36), and inserting c back again, we get finally

$$M_c v^i + \frac{1}{c^2} S^{ik} a_k = o, \tag{39}$$

or in vector form

$$M_c \hat{v} + \frac{1}{c^2} \hat{a} \times \hat{S} = o. \tag{39'}$$

These linear differential equations can be integrated without trouble, but the result may also directly be seen from (39′): the motion takes place in a plane perpendicular to the constant $\hat{S}$, the magnitude of the acceleration (perpendicular to $\hat{S}$) being constant and equal to

$$a = \frac{M_c c^2}{S_c}. \tag{40}$$

It is therefore a uniform motion along a circle with the angular velocity

$$\omega = \frac{M_c c^2}{S_c} = \frac{m_0 c^2}{S_0} \left(1 - \frac{v^2}{c^2}\right). \tag{41}$$

Here S_c denotes the constant magnitude of the angular momentum in the rest-system of the circle, an S_0 the same quantity in the rest-system of the particle; v is the scalar velocity in the rest-system of the circle. In the case now considered, in which $\hat{v}$ and $\hat{S}$ are perpendicular to one another, there exist a simple relation between S_c and S_0, following from (24) and (32).

$$S_c = \frac{S_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{42}$$

The radius of the circle is given by

$$r = \frac{S_c v}{M_c c^2} = \frac{S_0 v}{m_0 (c^2 - v^2)}. \tag{43}$$

Thus, in the specially chosen frame of reference Σ_c the motion of the free spin-particle is a uniform circular motion; in every other

inertial system the motion is a relativistic superposition of such a motion with a uniform translation.

7. To write down the equations of motion of the spin-particle in an electromagnetic field we must take into account the force acting on such a particle in an electromagnetic field as well as the torque exerted by the field on the particle, this last being the limit of the moment of all the forces acting on a small magnet when the size of the magnet tends to zero. We assume, of course, that the particle has an electric charge, ε, and a «purely magnetic» magnetoelectric moment, $\mu^{\alpha\beta}$, connected with the three-dimensional magnetic moment $\hat{\mu}$ and electric moment $\hat{\pi}$, as follows:

$$\{\mu^{23}, \mu^{31}, \mu^{12}\} = \hat{\mu}, \quad \{\mu^{14}, \mu^{24}, \mu^{34}\} = \hat{\pi}. \tag{44}$$

We know already that the moment is «purely magnetic» when

$$\mu^{\alpha\beta} u_\beta = o. \tag{45}$$

Let the electromagnetic field be given in the usual way by the bivector (skew-symmetrical tensor of rank two) $F_{\alpha\beta}$, connected with the magnetic intensity $\hat{H}$ and the electric intensity $\hat{E}$, as follows:

$$\{F_{23}, F_{31}, F_{12}\} = \hat{H}, \quad \{F_{14}, F_{24}, F_{34}\} = \hat{E}. \tag{46}$$

It may be noticed that the *contravariant* components of $\mu^{\alpha\beta}$ correspond to the components of $\hat{\mu}$ and $\hat{\pi}$ in the same manner as the *covariant* components of $F_{\alpha\beta}$ to $\hat{H}$ and $\hat{E}$; the potential energy of a rigid dipole in the field can be then expressed by the simple formula (with the same sign before each of the two terms on the right-hand side)

$$-\frac{1}{2} F_{\alpha\beta}\mu^{\alpha\beta} = -\hat{\mu} \cdot \hat{H} - \hat{\pi} \cdot \hat{E}. \tag{47}$$

We shall also assume that

$$\mu^{\alpha\beta} = \varkappa s^{\alpha\beta}, \tag{48}$$

where $\varkappa$ is a constant (not necessary equal to $\varepsilon / m_{00} c$).

The equation (25) expressing the law of conservation of linear momentum has to be supplemented on the right-hand side by two additional terms, the first being the well known expression for the Lorentz force

$$\frac{\varepsilon}{c} F_{\alpha\sigma} u^\sigma, \tag{49}$$

and the second

$$\frac{1}{2} \mu^{\rho\sigma} \partial_\alpha F_{\rho\sigma} \tag{50}$$

representing the relativistic generalization of the force acting on a magnetoelectric dipole in an electromagnetic field. The space components of the 4-vector (50) form a 3-vector

$$\nabla(\hat{\mu}\cdot\hat{H})+\nabla(\hat{\pi}\cdot\hat{E})=(\hat{\mu}\cdot\nabla)\hat{H}+(\hat{\pi}\cdot\nabla)\hat{E}+\hat{\mu}\times\mathrm{rot}\,\hat{H}+\hat{\pi}\times\mathrm{rot}\,\hat{E},\quad(51)$$

where only the vectors $\hat{H}$ and $\hat{E}$ (as space-time functions) but not $\hat{\mu}$ and $\hat{\pi}$ are subject to differentiation. In a magnetostatic field only the first term on the right-hand side remains; it represents then the usual expression of the «Stern-Gerlach force».[6]

Thus, from (25), we obtain

$$\boxed{\dot{G}_\alpha=\frac{\varepsilon}{c}F_{\alpha\sigma}u^\sigma+\frac{1}{2}\mu^{\rho\sigma}\partial_\alpha F_{\rho\sigma}.}\quad(52)$$

In like manner, equation (26) has to be supplemented by the four-dimensional expression of the torque exerted by the electromagnetic field on a magnetoelectric dipole, namely[7]

$$N_{\alpha\beta}=\mu_{\alpha\sigma}F^\sigma_{\beta}-\mu_{\beta\sigma}F^\sigma_{\alpha},\quad(53)$$

$$\{N_{23},N_{31},N_{12}\}=\hat{\mu}\times\hat{H}+\hat{\pi}\times\hat{E},\quad\{N_{14},N_{24},N_{34}\}=-\hat{\pi}\times\hat{H}+\hat{\mu}\times\hat{E}.\quad(54)$$

We get thus instead of (26)

$$\boxed{\dot{S}_{\alpha\beta}=G_\alpha u_\beta-G_\beta u_\alpha+\mu_{\alpha\sigma}F^\sigma_{\beta}-\mu_{\beta\sigma}F^\sigma_{\alpha}.}\quad(55)$$

In Section 3 we have deduced from (26) the expression (27) for the linear momentum, after having introduced the invariant mass m_0 by (28); this mass remains constant along the world-line of the particle in virtue of the equations of motion of a free particle. In an electromagnetic field, m_0 is not constant in general, but $\dot{m}_0$ may be expressed as the derivative with respect to the proper time τ of the expression $-F_{\rho\sigma}\mu^{\rho\sigma}/2c^2$; we can therefore introduce a new constant

$$m_{00}=-\frac{1}{c^2}G_\sigma u^\sigma+\frac{1}{2c^2}F_{\rho\sigma}\mu^{\rho\sigma}=m_0+\frac{1}{2c^2}F_{\rho\sigma}\mu^{\rho\sigma},\quad(56)$$

which reduces to m_0 in field-free space.

Multiplying (55) by u^β and taking into account (56), (45) and the equation resulting from (24) on differentiation with respect

[6] Cf., e. g., J. Frenkel, *Lehrbuch der Elektrodynamik II* (Berlin, 1926), p. 91.

[7] We follow the very convenient rule that — if not otherwise indicated by dots — the lower, covariant, indices have always to be considered as preceding the upper, contravariant, ones, for example $F_\alpha^{\beta}=F_{\alpha}^{\cdot\beta}$, $\Gamma^\gamma_{\alpha\beta}=\Gamma_{\alpha\beta}^{\cdot\cdot\gamma}$.

to τ, we get the following expression for the linear momentum of the particle in an electromagnetic field

$$\boxed{G_\alpha = \left(m_{00} - \frac{1}{2c^2}\mu_{\rho\sigma}F^{\rho\sigma}\right)u_\alpha + \frac{1}{c^2}S_{\alpha\sigma}\dot{u}^\sigma + \frac{1}{c^2}\mu_{\alpha\sigma}F^{\rho\sigma}u_\rho.} \tag{57}$$

The coefficient of u_α is the sum of the constant mas[8] m_{00} (not depending on the field) and an additional mass corresponding to the potential energy of the dipole in the electromagnetic field.

The equations (52), (55) and (56), together with the expression (57), which is a consequence thereof, are equivalent to Frenkel's equations (21) and (13a).

Equation (57) may be also written as

$$G_\alpha = m_{00}u_\alpha + \frac{1}{c^2}S_{\alpha\sigma}\dot{u}^\sigma - \frac{3}{2c^2}\mu_{[\rho\sigma}u_{\alpha]}F^{\rho\sigma}. \tag{57'}$$

8. Mathisson has noticed allready that the frequency (41) becomes identical with the frequency of Schrödinger's *Zitterbewegung* of a Dirac electron if we put $S_0 = h/2$ and m_0 equal to the mass of an electron. This agreement, however, is only valid in the non-relativistic case.

Still another detail points to the fact that the spin-particle as considered here, as well as by Thomas, Frenkel, Mathisson and others, cannot supply an adequate «classical» (non-quantum) model of an electron; namely, the radius (43) of the circle on which such a free particle moves can acquire arbitrary large, macroscopical, values, a result which is obviously contradicted by experiment.

References

(1) M. Mathisson, Acta Phys. Pol. **VI,** 163 (1937).
(2) J. Lubański, Acta Phys. Pol. **VI,** 356 (1937).
(3) J. Frenkel, Zeits f. Physik **37,** 243 (1926).
(4) M. Mathisson, Acta Phys. Pol. **VI,** 218 (1937).

[8] The proof of the constancy of m_{00} along the world line of the particle will be given in a subsequent paper. This proof will be based among other things on the proportionality of $\mu^{\alpha\beta}$ to $s^{\alpha\beta}$ given by (48).

RELATIVISTIC DYNAMICS OF SPIN-PARTICLES MOVING WITH THE VELOCITY OF LIGHT*

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1. In the preceding paper we have deduced the equations of motion of a spin-particle moving with a velocity v smaller than that of light, c. We shall now consider the equations of motion of a spin-particle moving with the velocity of light. Though this last case may be conceived in a certain sense, which will be specified later, as a limiting case of the former, it presents many distinctive features and, in any case, it is not simply the limiting case of the former when v tends to c at constant proper mass m_0 and constant proper angular momentum[1] s_0. From the point of view of the theory of relativity it constitutes by itself an interesting and hitherto not contemplated example of a set of differential equations leading to a curvilinear motion with the velocity of light; however, the most interesting feature of such a motion is apparently its close analogy with the behaviour of Dirac's quantum-mechanical electron.

2. Henceforth we shall refer to the motion of a spin particle with a velocity smaller than that of light as the *first case;* by *second case* we shall understand the motion of a spin particle with a velocity changing in direction but having invariably the same magnitude c. To obtain the equations of motion of a particle in the second case in a four-dimensional form, we must, first of all, change the parameter along the world-line of the particle from τ, the proper time, to an arbitrary new parameter p, leaving the world-line of the particle unchanged, and only afterwards, as the second step of the reasoning, we may distort the world-line in such a way as to make it everywhere tangent to the light-cone in the corresponding world-point. The world-line will then represent a motion of the particle with the velocity of

* See the preceding paper (henceforth designated by I), references (*) and (**).

[1] From now on we shall denote the angular momentum of the particle by $s^{\alpha\beta}$ and s, rather than by $S^{\alpha\beta}$ and S as in I where small letters where reserved for denoting physical properties of the spin-fluid.

light. The proper time does not flow any more on such a line, and we were therefore compelled to change the parameter from τ to p. The new arbitrary parameter p is subject only to the restriction that it should grow in the direction of growing t (in any allowed coordinate-system). It satisfies therefore the relation

$$\tau' \geqslant 0, \tag{1}$$

the equality being valid only on isotropic world-lines, i. e., in our second case; primes denote differentiation with respect to p.

In the first case τ itself is a special case of p. When passing from one parameter to another, i. e., changing the parametrization on the world-line of the particle from p to $\bar{p}$, the condition

$$\frac{dp}{d\bar{p}} > 0 \tag{2}$$

must be satisfied.

3. We introduce now the 4-vector

$$w^\alpha = \frac{dx^\alpha}{dp} \tag{3}$$

as a generalization of the four-dimensional velocity, which does not exist in the second case. w^α is a «4-vector depending on the parametrization»; its four components transform like components of a 4-vector when the coordinates are transformed without change of parametrization, but they are all four multiplied by a common factor $dp/d\bar{p}$ when that parametrization is changed. The 4-vector w^α may therefore play the role of an auxiliary mathematical quantity, but it cannot have any direct physical meaning.

In the first case,

$$w^\alpha = \tau' u^\alpha \tag{4}$$

and

$$w_\alpha w^\alpha = u_\alpha u^\alpha \tau'^2 = -c^2 \tau'^2, \tag{5}$$

where u^α is the four-dimensional velocity of the particle.

In the second case, the world-line of the particle is isotropic,[2] $\tau' = 0$, and the four-dimensional velocity u^α does not exist any more, as it components are either infinite or indeterminate. But w^α retains obviously a definite meaning and satisfies the relation

$$w_\alpha w^\alpha = 0. \tag{6}$$

[2] In the usual representation of the Minkowski space ($x^1, x^2, x^3, x^4 = ct$) it is a curve which is everywhere inclined at an angle of 45^0 to the hyperplane $x^4 = 0$.

We can thus characterize w^α as an isotropic vector depending on the parametrization on the world-line of the particle but always tangent to that line.

4. If we change τ into p in the equations of motion I (24)—(26), take into account equation (4), and multiply all the equations by τ', we get

$$\boxed{\begin{aligned} &G'_\alpha = o, \\ &s'_{\alpha\beta} = G_\alpha w_\beta - G_\beta w_\alpha, \\ &s^{\alpha\beta} w_\beta = o. \end{aligned}} \qquad \begin{aligned}(7)\\(8)\\(9)\end{aligned}$$

Because of the homogeneity of the equations I (24)—(26) in the derivatives with respect to τ, τ' has disappeared from (7)—(9), and equations (7)—(9) are valid not only in the first case (as alternative expressions of the same equations of motion as I (24)—(26)), but also in the second case, when $\tau' = o$.

Some interesting complications arise only when we set out to transform in the same manner equation I (27). Taking into account (4), (9) and I (28), we get then

$$\tau' G^\alpha = m w^\alpha + \frac{1}{c^2} s^{\alpha\sigma} w'_\sigma, \tag{10}$$

where we have put, in analogy to I (28),

$$\boxed{m = -\frac{1}{c^2} w_\sigma G^\sigma.} \tag{11}$$

m is «a scalar depending on the parametrization».[3] Equation I (27) is no more homogeneous in the derivatives with respect to τ, and therefore the equation for the second case, which we get from it by putting $\tau = o$, differs in form from the original equation. We find namely

$$\boxed{m w^\alpha + \frac{1}{c^2} s^{\alpha\sigma} w'_\sigma = o.} \tag{12}$$

[3] It is different from zero when G^α is a time-like vector. In the first case, when w^α is also a time-like vector, this result is obvious. In the second case, when w^α is isotropic, our assertion is a consequence of the following often useful theorem: Any 4-vector perpendicular to an isotropic 4-vector and not parallel to it is space-like (of course, only real 4-vectors are taken into consideration). If we require as well that for $s^{\alpha\beta}$ tending to zero all our equations should go over into the ordinary equations of spin-less dynamics, then m will be positive.

Though G^α disappeared from (12), the equations of motion of an unconstrained spin-particle moving with the velocity of light can be generally solved, and give a result similar to that obtained in the first case.

As in I, equation (12) is a consequence of (8) and (9), together with equation (6) which expresses the fact that the scalar velocity of the particle is c. This can be shown by multiplying (8) by w^β, taking into account equation (6) and equation (9) once differentiated. *Vice versa,* from (13) and (9) we can deduce (6) by multiplying (12) by w^α. Thus, if we consider (7), (8), (9) and (12) as the equations of motion of a free spin-particle, they will imply as a consequence that the velocity of the particle is equal to the velocity of light. Moreover, this conclusion is independent of equation (7), and it will therefore prove also correct for a spin-particle in an electromagnetic field, for which equations (8), (9) and (12) remain unchanged (see Section 7 below).

5. *The time as parameter.* We may avail ourselves of the arbitrariness in the choice of the parameter to give to the equations of motion a new interesting form. Let us agree to put p equal to $t = x^4/c$ in each allowed coordinate-system; this amounts to changing the parametrization at every change of the system of coordinates in such a way that p should remain always equal to t in the coordinate system of interest. Then w^α becomes

$$v^\alpha = \frac{dx^\alpha}{dt} = \{\hat{v}|c\}. \tag{13}$$

As it is well known, the four quantities v^α do not form a 4-vector, neither in the ordinary sense, nor as a «4-vector depending on the parametrization»; they form instead a new «geometrical object» in M_4, which we shall call «pseudovector». The components of a pseudovector would transform like components of a vector but for an additional factor $dt/d\bar{t} = dx^4/d\bar{x}^4$. The magnitude of a pseudovector, for example $\sqrt{v_\alpha v^\alpha}$, or its scalar product by a vector, for example v G_α are pseudoscalars: they are multiplied by $dx^4/d\bar{x}^4$ when the coordinates are transformed from one inertial frame of reference to another.

The equations of motion of a spin-particle moving with the velocity of light take now the form

$$G'_\alpha = 0, \tag{14}$$

$$s'_{\alpha\beta} = G_\alpha v_\beta - G_\beta v_\alpha, \tag{15}$$

$$s^{\alpha\beta} v_\beta = 0. \tag{16}$$

Though v^α is not a vector, its components have simple physical meanings. The equations (14)—(16) are interesting in so far, as they give an invariant description of the motion though they are not tensor equations.

Just as before, (15) multiplied by v^β and (16) once differentiated with respect to t yield

$$m^* v^\alpha + \frac{1}{c^2} s^{\alpha\sigma} a_\sigma = o, \tag{17}$$

where

$$a^\sigma = \frac{dv^\sigma}{dt} = \{\hat{a} \mid o\} \tag{18}$$

and

$$m^* = -\frac{1}{c^2} G_\sigma v^\sigma \tag{19}$$

is a pseudoscalar; $\hat{a}$ is the three-dimensional acceleration vector.

6. *General Solution of the Equations of Motion of a Free Spin-Particle Moving with the Velocity of Light.* Obviously, a rest-system, that is, an inertial system of reference in which the particle is momentary at rest, does not exist for a particle moving with the velocity of light, nevertheless certain inertial frames of reference stand out from among all others, those namely — we shall denote them by Σ_c — in which the space components of G^α vanish,[4]

$$G^k \overset{c}{=} o. \tag{20}$$

Any two Σ_c-systems (referring to the same position of the particle on its world-line) differ only by a space-rotation without change of the time-coordinate. We shall call them «rest-systems of the circle», as the particle moves in each of them on a stationary circle, a result which we are going now to prove.

It will be found advantageous to use the formulae of the preceding section (with t as parameter). Then, $a^4 = o$, by (18), and we may write (17) in three-dimensional vector-form, as follows:

$$m^* \hat{v} + \frac{1}{c^2} \hat{a} \times \hat{s} = o. \tag{21}$$

In Σ_c, m^* is a constant, as

$$m^* \overset{c}{=} \frac{1}{c} G^4 \tag{22}$$

[4] We have to assume, of course, that G^α is a time-like vector.

by (19) and (20), and G^a is a constant for a free particle. We can also put

$$m^* \stackrel{c}{=} M_c \tag{23}$$

in Σ_c, if we define M_c, as in I, by the relation

$$M_c^2 = -\frac{1}{c^2} G_\sigma G^\sigma. \tag{24}$$

Thus, we can write (21) in the form

$$M_c \hat{v} + \frac{1}{c^2} \hat{a} \times \hat{s} \stackrel{c}{=} o. \tag{25}$$

As in I, the final result is directly evident from (25) if we remark that

$$\hat{s} \equiv \{s^{ik}\} \stackrel{c}{=} \mathrm{const} \tag{26}$$

as a consequence of (15) and (20). In the Σ_c-frame of reference the particle revolves uniformly on a stationary circle with the angular velocity

$$\omega_c = \frac{M_c c^2}{s_c}, \tag{27}$$

the linear velocity beeing, of course, c. This formula is identical with the first form of I (41); the two last forms given in I have no counterpart here, as m_0 and s_0 do not exist in the present case.

The radius of the circle in Σ_c — which we may call, as in I, «proper radius of the circle» — is now given by

$$r_c = \frac{s_c}{c M_c}. \tag{28}$$

It coincides with the first form of I (43) for $v = c$.

7. Let us now approach the question of the motion of our spin-particle in an electromagnetic field. The equations of motion I (52) and I (55) may be treated in the manner indicated in Section 2. It may be considered as a lucky circumstance that these equations get thereby greatly simplified: the last term in I (52) and the two last in I (55) get in the first step of the operation the factor τ' (or, more precisely, all the other terms the factor $1/\tau'$) and vanish therefore at the second step. There remains only one additional term depending on the electromagnetic field, which is analogous to the well known four-dimensional expression of the Lorentz force, the only difference being that w^α superseded the four-dimensional velocity u^α.

Finally, the equations of motion of a spin-particle moving with the velocity of light in an electromagnetic field given by the bivector $F_{\alpha\beta}$ are

$$\boxed{\begin{aligned} G'_\alpha &= \frac{\varepsilon}{c} F_{\alpha\beta} w_\beta && (29)\\ s'_{\alpha\beta} &= G_\alpha w_\beta - G_\beta w_\alpha, && (30)\\ s_{\alpha\beta} w^\beta &= o. && (31)\end{aligned}}$$

8. Multiplying (30) by $s^{\alpha\beta}$ and making use of (31) we obtain

$$s'_{\alpha\beta}\, s^{\alpha\beta} = o, \tag{32}$$

hence $s_{\alpha\beta}\, s^{\alpha\beta}$ is a constant. But $s_{\alpha\beta}\, s^{\alpha\beta}$ vanishes in a field-free space, as may be easily verified[5] from the results of Section 6, and therefore, in consequence of (32), it vanishes everywhere. Thus,

$$s_{\alpha\beta}\, s^{\alpha\beta} = o. \tag{33}$$

Further particulars will be given in a following paper.

[5] It is sufficient to remark that in the circular motion in Σ_c, $\hat{v}$ and $\hat{s}$ are perpendicular, and therefore $q^2 = s^2$, in consequence of the relation $\hat{q} = \frac{1}{c}\, \hat{v} \times \hat{s}$, v being equal to c. We get then $\frac{1}{2} s_{\alpha\beta}\, s^{\alpha\beta} = \hat{s} \cdot \hat{s} - \hat{q} \cdot \hat{q} = s^2 - q^2 = o$. Incidentally it may be noticed that by following the same argument back again, we can infer from (33) that $\hat{v}$ is always perpendicular to $\hat{s}$, not only for a free particle, but also for a particle in an electromagnetic field. It is not so in the first case.

FURTHER CONTRIBUTIONS TO THE DYNAMICS OF SPIN-PARTICLES MOVING WITH A VELOCITY SMALLER THAN THAT OF LIGHT*

By Jan WEYSSENHOFF, Institute of Theoretical Physics, Jagellonian University, Kraków.

1. In a preceding paper written with the late Mr. A. Raabe I worked out the equations of motion of a particle endowed with spin, by considering at first the relativistic equations of motion of a spin fluid, and passing then to the limiting case of an infinitely small portion of such a fluid with infinitely large mass-density and angular-momentum density. These equations are equivalent to special cases of equations previously found by Frenkel and by Mathisson, though the form of our equations is much simpler, due to the explicit introduction of the 4-vector of linear momentum and energy G^{α}. Here, I shall outline still another method of approach to the same equations, a method which is, to be sure, not so correct in principle as that of Mathisson, but it may prove suggestive by its conciseness and its elementary character.

2. Let us consider a particle possessing a velocity $\hat{v}$ and a linear momentum $\hat{G}$, and let us assume the validity of the laws of conservation of linear momentum and of angular momentum, the only departure from classical treatment being that we do not assume *a priori* any relation between $\hat{G}$ and $\hat{v}$. We are therefore compelled to introduce the law of conservation of angular momentum (in a generalized form with outer and inner angular momentum) independently of the law of consevation of linear momentum.

The law of conservation of linear momentum for a free particle may be expressed as follows:

$$\hat{G}' = o, \tag{1}$$

the prime denoting differentiation with respect to the time. If we define in the usual manner the moment of momentum (external

* See I, reference (*). The two preceding papers will be designated henceforth by I and II.

angular momentum) as the vector-product of the radius-vector $\hat{r}$ by $\hat{G}$ and differentiate it with respect to the time, we get

$$\frac{d}{dt}(\hat{r}\times\hat{G})=\hat{r}\times\hat{G}'+\hat{v}\times\hat{G}=\hat{v}\times\hat{G}. \tag{2}$$

Thus, if $\hat{G}$ is not parallel to $\hat{v}$, the moment of momentum is not a constant of motion, as in spinless dynamics, but we can restore the validity of the law of conservation of angular momentum by introducing a spin, or internal, angular momentum $\hat{s}$, in addition to the orbital, or external, angular momentum $\hat{l}=\hat{r}\times\hat{G}$. To this end, we put

$$\hat{s}'=\hat{G}\times\hat{v}, \tag{3}$$

and (20) goes over into

$$\frac{d}{dt}(\hat{l}+\hat{s})=o, \tag{4}$$

that is, into the generalized law of conservation of angular momentum.

3. Now, to obtain the Frenkel-Mathisson equations (in special relativity) we need only translate the equations (1) and (3) into four-dimensional tensor-form. Let $G^{\alpha}=\{\hat{G}, W/c\}$ be the momentum-energy 4-vector and $s^{\alpha\beta}$ the spin bivector (antisymmetrical tensor of rank two). Of the two three-dimensional vectors

$$\hat{s}=\{s^{23}, s^{31}, s^{12}\}, \quad q=\{s^{14}, s^{24}, s^{34}\}, \tag{5}$$

the former is the three-dimensional spin-vector, and the latter may be considered merely as an auxiliary mathematical concept, as it can be always eliminated by making use of the following relation

$$\hat{q}=\frac{1}{c}\hat{v}\times\hat{s}, \tag{6}$$

which is the three-dimensional expression of the four-dimensional relation

$$s^{\alpha\beta}u_{\beta}=o. \tag{7}$$

This last equation — used also by F r e n k e l and by M a t h i s s o n — may be considered as the condition of non-existence of negative mass, as in the corresponding electromagnetic case

$$\mu^{\alpha\beta}u_{\beta}=o \tag{8}$$

is the condition that the magneto-electric dipole-moment $\mu^{\alpha\beta}$ should be «purely magnetic», that is, that the electric moment $\hat{\pi}=\{\mu^{14}, \mu^{24}, \mu^{34}\}$

should vanish in the coordinate system in which the particle momentary rests ($u^k = o$, $k = 1, 2, 3$).

Incidentally, it may be noticed that

$$\mu^{[\alpha\beta} u^{\gamma]} = o \tag{9}$$

is the condition that $\mu^{\alpha\beta}$ should be «purely electric», i. e., that the magnetic moment $\hat{\mu} = \{\mu^{23}, \mu^{31}, \mu^{12}\}$ should vanish in the rest-system of the particle.

Obviously, the four-dimensional generalizations of (1) and (3) are now

$$\dot{G}^\alpha = o, \tag{10}$$

$$\dot{s}^{\alpha\beta} = G^\alpha u^\beta - G^\beta u^\alpha, \tag{11}$$

where the dot signifies differentiation with respect to the proper time τ.

As in I, (11) multiplied by u_β yields

$$G^\alpha = m_0 u^\alpha + \frac{1}{c^2} s^{\alpha\sigma} \dot{u}_\sigma, \tag{12}$$

where

$$m_0 = -\frac{1}{c^2} G_\sigma u^\sigma. \tag{13}$$

Multiplying (12) by $\dot{u}^\alpha$, we get

$$G^\alpha \dot{u}_\alpha = o, \tag{14}$$

which shows that the momentum-energy vector G^α, which ceased to be parallel to the four-dimensional velocity u^α, remains four-dimensionally orthogonal to the four-dimensional acceleration $\dot{u}^\alpha$, just as u^α

From (11) and (14) we get

$$\dot{s}^{\alpha\beta} \dot{u}_\beta = o. \tag{15}$$

4. The same method applies also to a spin-particle (with charge ε and magneto-electric moment $\mu^{\alpha\beta}$) moving in an electromagnetic field $F_{\alpha\beta}$, the only difference being that in the three-dimensional argument of Section 2 we must also take into account the force $\hat{F}$ and the torque $\hat{N}$ exerted by the electromagnetic field on the particle. Equation (1) becomes

$$\hat{G}' = \hat{F}, \tag{16}$$

and the intermediate equation (2) takes the form

$$\frac{d}{dt}(\hat{r} \times \hat{G}) = \hat{r} \times \hat{F} + \hat{v} \times \hat{G}. \tag{17}$$

To restore the validity of the law of conservation af angular momentum, we must put therefore, instead of (3),

$$\hat{s}' = \hat{G} \times \hat{v} + \hat{N}, \tag{18}$$

and equation (4) takes the form

$$\frac{d}{dt}(\hat{l} + \hat{s}) = \hat{r} \times \hat{F} + \hat{N}. \tag{19}$$

The translation of these equations into tensor language and the four-dimensional generalizations of $\hat{F}$ and N were given in I, Section 7, where eventually the following equations have been obtained:

$$\dot{G}_\alpha = \frac{\varepsilon}{c} F_{\alpha\sigma} u^\sigma + \frac{1}{2} \mu^{\rho\sigma} \partial_\alpha F_{\rho\sigma}, \tag{20}$$

$$\dot{s}_{\alpha\beta} = G_\alpha u_\beta - G_\beta u_\alpha + \mu_{\alpha\sigma} F^\sigma_\beta - \mu_{\beta\sigma} F^\sigma_\alpha, \tag{21}$$

$$G_\alpha = (m_{00} - \frac{1}{2c^2} \mu^{\rho\sigma} F_{\rho\sigma}) u_\alpha + \frac{1}{c^2} s_{\alpha\sigma} \dot{u}^\sigma + \frac{1}{c^2} \mu_{\alpha\sigma} F^{\rho\sigma} u_\rho, \tag{22}$$

with

$$m_{00} = -\frac{1}{c^2} G_\sigma u^\sigma + \frac{1}{2c^2} F_{\rho\sigma} \mu^{\rho\sigma}. \tag{23}$$

In (22) we may also put together the second and the fourth terms on the right-hand side, and write

$$G_\alpha = m_{00} u_\alpha + \frac{1}{c^2} s_{\alpha\sigma} \dot{u}^\sigma + \frac{3}{2c^2} \mu_{[\alpha\sigma} u_{\rho]} F^{\rho\sigma}. \tag{24}$$

5. To prove the constancy of m_{00} we may proceed as follows. By differentiating equation (23) with respect to the proper time τ, we get $(c = 1)$

$$\dot{m}_{00} = -\dot{G}_\sigma u^\sigma - G_\sigma \dot{u}^\sigma + \frac{1}{2} \dot{\mu}^{\rho\sigma} F_{\rho\sigma} + \frac{1}{2} \mu^{\rho\sigma} \dot{F}_{\rho\sigma}. \tag{25}$$

Equation (20) miltiplied by u^α yields

$$\dot{G}_\alpha u^\alpha = \frac{1}{2} \mu^{\rho\sigma} \dot{F}_{\rho\sigma}, \tag{26}$$

as

$$\dot{F}_{\rho\sigma} = u^\alpha \partial_\alpha F_{\rho\sigma}. \tag{27}$$

Thus the first and the last terms in (25) cancel one another. The same happens with the two remaining terms if we assume the proportionality of $\mu^{\alpha\beta}$ to $s^{\alpha\beta}$, that is, the relation

$$\mu^{\alpha\beta} = \varkappa\, s^{\alpha\beta} \tag{28}$$

with an arbitrary value of $\varkappa$ (not necessarily equal to $\varepsilon/m_{00}c$). In fact, from (22) we get

$$G_\alpha \dot{u}^\alpha = \mu_{\sigma\alpha} F^{\alpha\rho} u_\rho \dot{u}^\alpha, \tag{29}$$

and multiplying (21) by $\frac{1}{2} F^{\alpha\beta}$ and (22) by $u_\beta F^{\alpha\beta}$, we get

$$\frac{1}{2} F^{\alpha\beta} \dot{s}_{\alpha\beta} = \dot{G}_\alpha u_\beta F^{\alpha\beta} = s_{\sigma\alpha} F^{\sigma\rho} u_\rho \dot{u}^\alpha. \tag{30}$$

The comparison of the two preceding equations leads to the result stated above, and hence to

$$\dot{m}_{00} = 0, \tag{31}$$

Q. E. D.

6. In the dynamics of spin-particles the velocity $\hat{v}$ of a particle points in general in another direction than its momentum $\hat{G}$. However, we can formally introduce a new vector $\hat{V}$ connected in the same way with $\hat{G}$ as the velocity is connected with the linear momentum in the old spinless dynamics. We may call then $\hat{v}$ (as defined by the world-line of the particle) the «kinematical velocity» and $\hat{V}$ (as defined by the momentum of the particle) the «dynamic velocity» (though Eddington uses the same terms in another meaning.

We define $\hat{V}$ by the equation

$$M\hat{V} = \hat{G} = (G^1, G^2, G^3) \tag{32}$$

where

$$M = \frac{1}{c} G^4. \tag{33}$$

Alternatively, $\hat{v}$ might be called «velocity of the particle» and $\hat{V}$ «velocity of the circle»; similarly M might be called «mass of the circle» and M_c introduced by the relation

$$-G_\sigma G^\sigma = M_c c^2 \tag{34}$$

«rest-mass of the circle», as it is the value of M in the «rest-system

of the the circle» (in which $\hat{G} = o$).[1] It is connected with M by the following relation, which is a direct consequence of (33) and (34)

$$M = \frac{M_c}{\sqrt{1 - V^2/c^2}} \tag{35}$$

Taking into account (32), (33) and (35) we get from (13) the following relation between m_0 and M_c

$$m_0 = \frac{M}{\sqrt{1 - v^2/c^2}\,\sqrt{1 - V^2/c^2}}\left(1 - \frac{\hat{v} \cdot \hat{V}}{c^2}\right). \tag{36}$$

Obviously, all the formulae of this section are independent of the relation existing between $\hat{v}$ and $\hat{V}$ as they are merely the consequence of our initial assumption of the validity of the energy-momentum principle.

7. This section contains a list of the equations of the dynamics of spin-particles in three-dimensional vector form. To simplify matters c has been throughout put equal to unity.[2] The numbers on the left point to the four-dimensional tensor equations from which the corresponding vector equations have been derived. γ stands for $1/\sqrt{1 - v^2}$.

(7) $$\hat{q} = \hat{v} \times \hat{s}, \tag{37}$$

(8) $$\hat{\pi} = \hat{v} \times \hat{\mu}, \tag{38}$$

(20) $$\hat{G}' = \varepsilon(\hat{E} + \hat{v} \times \hat{H}) + \sqrt{1 - v^2}\{\nabla(\hat{\mu} \cdot \hat{H}) + \nabla(\hat{\pi} \cdot \hat{E})\}, \tag{39}$$

(20) $$W' = M' = \varepsilon\hat{E} \cdot \hat{v} + \sqrt{1 - v^2}\left\{\hat{\mu} \cdot \frac{\partial \hat{H}}{\partial t} + \hat{\pi} \cdot \frac{\partial \hat{E}}{\partial t}\right\}, \tag{40}$$

(21) $$\hat{s}' = \hat{G} \times \hat{v} + \sqrt{1 - v^2}\{\hat{\mu} \times \hat{H} + \hat{\pi} \times \hat{E}\}, \tag{41}$$

(21) $$\hat{q}' = \hat{G} - M\hat{v} + \sqrt{1 - v^2}\{\hat{\pi} \times H - \hat{\mu} \times \hat{E}\}, \tag{42}$$

(24) $$\hat{G} = m_{00}\gamma\hat{v} + \gamma^2\hat{a} \times \hat{s} - \gamma(\hat{v} \cdot \hat{\mu})\hat{H} + \gamma(\hat{\mu} + \hat{v} \times \hat{\pi}) \times \hat{E}, \tag{43}$$

(24) $$W = M = m_{00}\gamma - \gamma^2\hat{a} \cdot \hat{q} - \frac{1}{\gamma}\hat{\mu} \cdot \hat{H} - \gamma(\hat{v} \cdot \hat{\mu})(\hat{v} \cdot \hat{H}), \tag{44}$$

$$m_0 = \gamma(M - \hat{G} \cdot \hat{v}), \tag{45}$$

(23) $$m_{00} = m_0 + \hat{\mu} \cdot \hat{H} + \hat{\pi} \cdot \hat{E} = \text{const.} \tag{46}$$

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[1] These denominations are misleading in so far as $\hat{G}$ was assumed from the outset to be the momentum and $cG^4 = Mc^2$ the energy *of the particle*.

[2] Its restoration is quite easily accomplished if we bear in mind the following dimensional equations: $W = Gv = m\,c^2 = s' = q' = \mu H = \mu E = \pi E = \pi H = as/c$ and $\varepsilon E = \varepsilon H = G'$.

From (42) and (37) we can draw still another relation between $\hat{G}$ and $\hat{v}$, namely

$$\hat{G} = M\hat{v} + \hat{a} \times \hat{s} + \hat{v} \times \hat{s}' - \sqrt{1 - v^2}\,\{\hat{\pi} \times \hat{H} - \hat{\mu} \times \hat{E}\}. \tag{47}$$

It is equivalent to the previously found relation (43) thanks to (41), (45) and (46).

Equation (45) is only another form of equation (36).

The second term on the right-hand side of (44), which may be written in the form

$$\frac{[\hat{s}\,\hat{v}\,\hat{a}]}{c^2 - v^2},$$

has been called by Mathisson «acceleration energy» *(Beschleunigungsenergie)*. For a free particle in the rest-system of the circle it is constant and negative.

8. The general solution of the equations of motion of a free particle is directly apparent from (47), or from (43) together with (45): in a frame of reference in which $\hat{G}$ vanishes

$$M_c\hat{v} + \frac{1}{c^2}\hat{a} \times \hat{s} = o, \tag{48}$$

just as in I (39). The motion is then a uniform circular motion in a plane perpendicular to $\hat{s}$ (which is constant); the radius and the angular velocity were given in I. The sense of the motion on the circle is such that the moment of the velocity with respect to the center of the circle points in a direction opposite to that of $\hat{s}$.

In any other inertial system of reference the path may be described as a distorted screw-motion, if it does not happen to be plane. In the latter case $\hat{s}$, which remains perpendicular to the plane of motion, has a constant direction, but its magnitude is variable according to the equation

$$s = \frac{s_0}{\sqrt{1 - v^2/c^2}}, \tag{49}$$

following from $s^2 - q^2 = s_0^2$ and (6) when $\hat{v}$ and $\hat{s}$ are perpendicular. The path reminds of a contracted or elongated cycloid, according to whether the velocity of the center of the circle is smaller or larger than v, the velocity of the particle on the circle in the rest-system of the latter.

When the proper radius of the circle is infinitesimal, the motion of the particle is the same as for a spinless particle with a superimposed infinitesimal wave-like motion.

9. It was already shown that m_{00} is a constant of motion, so is the magnitude of $s^{\alpha\beta}$, i. e. the square root of

$$\tfrac{1}{2}\, s^{\alpha\beta}\, s^{\alpha\beta} = s^2 - q^2 = s_0{}^2 = \text{const.} \tag{50}$$

This becomes evident if we multiply (21) by $s^{\alpha\beta}$ and take into account (7) and (28).[3]

We may also infer from (7) that

$$s^{[\alpha\beta}\, s^{\gamma\delta]} = 0, \tag{51}$$

which is the condition of flatness[4] of the bivector $s^{\alpha\beta}$; in fact, the left-hand side expression in (51) is the square root of the determinant of the four homogeneous equations (7).

10. We know already that our equations of motion are equivalent to the Frenkel-Mathisson equation I (29), which is a differential equation of the third order for the (x^α)'s.[5] Therefore, to determine the motion of a spin-particle not only its initial position and velocity should be given, but also its initial acceleration; this departs so much from all one may expect of the behaviour of a material particle that we are drawn to the conclusion that not the singularity of the gravitational field itself, but only its mean position — or, in our case, the whole circle on which it moves — has to be considered as representing the material particle. This conception will be followed more in detail in the two following papers.

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[3] Notice that $\mu_{\alpha\sigma}\, s^{\alpha\beta}\, F^{\sigma}_{\beta} = \varkappa\, s_{\alpha\sigma}\, s^{\alpha\beta}\, F^{\sigma}_{\beta} = \varkappa\, s^{\sigma}_{\alpha}\, s^{\alpha\beta}\, F_{\beta\sigma} = 0$, as in the last expression the antisymmetric tensor $F_{\beta\sigma}$ is multiplied by a symmetrical one.

[4] I propose to call a bivector flat (rather than simple, which is the expression used by S c h o u t e n) if it can be represented as an alternating product of two vectors. A general bivector in a four-dimensional Minkowski space can be represented as a sum of two flat bivectors in two completely perpendicular planes.

[5] P. A. M. D i r a c has also considered an equation of the third order for the motion of an electron, but his equation (as well as the whole problem of the radiation of an electron which led to it) was entirely different. The additional term of the third order in his equation may be characterized as longitudinal, whereas our additional term is transversal. To avoid all sorts of physically meaningless motions D i r a c was compelled to introduce a strange restriction concerning the acceleration for t tending to infinity; this cannot be done in our case.

FURTHER CONTRIBUTIONS TO THE DYNAMICS OF SPIN-PARTICLES MOVING WITH THE VELOCITY OF LIGHT*

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1. In two previous papers written with the late Mr. A. Raabe[1] we deduced among other things the relativistic equations of motion of a spin-particle moving with the velocity of light. First, in I, we found the equations of motion of a spin-particle moving with a velocity smaller than that of light by integrating the equations of motion of a spin-fluid over an infinitesimal volume of that fluid. Secondly, in II, we changed the parameter along the world-line of the particle from τ, the proper time of the particle, to an arbitraty parameter p, leaving the world line of the particle unaltered, and distorted afterwards the world line in such a way as to make it everywhere tangent to the corresponding light-cone.

The same results may be also obtained in a simpler way. Instead of passing through the dynamics of a spin-fluid, we may write directly the relativistic equations of motion of a spin-particle by translating into four-dimensional tensor language the three-dimensional expressions of the laws of conservation of linear and angular momentum. This was done in III, Section 3. The second part of the above reasoning remains unaltered as in II.

2. For a free spin-particle the equations referred to above are

$$G^{\cdot\alpha} = 0, \tag{1}$$

$$s^{\cdot\alpha\beta} = G^{\alpha} w^{\beta} - G^{\beta} w^{\alpha}, \tag{2}$$

together with the condition

$$s^{\alpha\beta} w_{\beta} = 0 \tag{3}$$

expressing the non-existence of negative mass (see III, Section 3). The notations used are the same as in II.

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* See I, reference (*). The three preceding papers will be designated henceforth by I, II and III.

[1] I and II.

Multiplying (2) by w_β, and taking into consideration that $w_\alpha w^\alpha = o$, we get

$$m w^\alpha + \frac{1}{c^2} s^{\alpha\beta} w'_\beta = o, \tag{4}$$

where

$$m = -\frac{1}{c^2} G_\alpha w^\alpha. \tag{5}$$

In the corresponding equation I (27) referring to the *first case* [2], the right-hand member of (4) was equal to G^α, and I (27) gave the four-dimensional expression of the relation between G^α and u^α. Here G^α disappeared from (4)[3] and now it does not seem possible to eliminate G^α from the equations (1)—(4), and thus to get a differential equation of the third order for x^α corresponding to Mathisson's equation I (29). However, the equations of motion of a free particle can be solved in a similar way, and the results are very much the same, as was shown in II, Section 6.

3. As we know from II, Section 7, the equations (2) and (3) remain valid for a particle in an electromagnetic field, and only the equation (1) is changed and goes over into

$$G'_\alpha = \frac{\varepsilon}{c} F_{\alpha\beta} w^\beta. \tag{6}$$

This is an important simplification in comparison with the first case, in which not only a rather complicated term appears in the equation corresponding to (6) but also equation (2) has a less simple form.

4. It was shown in I and III that in the first case two different masses may be introduced which become identical when the spin vanishes (as a consequence of the fact that the 4-vector of linear momentum and energy is then proportional to the four-dimensional velocity). Here, in the *second case*, only one of those masses, namely

$$M_c = \sqrt{-\frac{1}{c^2} G_\alpha G^\alpha} \tag{7}$$

may have a physical significance, as the second one, defined as $-G_\alpha w^\alpha / c^2$ is a «scalar depending on the parametrization» and $G_\alpha u^\alpha$ is infinite.

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[2] Cf. II, Section 2.

[3] But nevertheless there exist in the second case also a three-dimensional relation between $\hat{G}$ and $\hat{v}$ (see eq. (16) below). In the first case there are two equivalent relations between the linear momentum and the velocity of the spin-particle, a four-dimensional one: I(27), and a three-dimensional one: III(43) or III(47).

It is obvious that the relations between G^α, M, M_c and V referred to in III, Section 6 apply also in the present case, as they are only dependent upon the laws of conservation of momentum and energy on which both our theories have been founded.

5. This Section contains a list of the equations of the dynamics of spin-particles moving with the velocity of light, in three-dimensional vector form. These equations are here given not in their most general form, but for $p = t$; the use of the time t as parameter was considered in II, Section 5. w^α a «vector depending on the parametrization» is then replaced by the «pseudovector» v^α, and the three-dimensional formulae take a simpler form, as $v^4 = c$ and $v'^4 = o$; moreover, the components of the pseudovector v^α possess a direct physical meaning. The numbers in parantheses written on the left of each of the equations below refer to the four-dimensional equation from which the corresponding three-dimensional one has been derived.

(1) $$\hat{G}' = \varepsilon\hat{E} + \frac{1}{c}\hat{v} \times \hat{H}, \tag{8}$$

(1) $$W' = \varepsilon\hat{E} \cdot \hat{v}, \tag{9}$$

(2) $$\hat{s}' = \hat{G} \times \hat{v}, \tag{10}$$

(2) $$\hat{q}' = c\,(G - M\hat{v}) = Mc\,(\hat{V} - \hat{v}), \tag{11}$$

where $M = G^4/c$ as in III (33),

(3) $$\hat{q} = \frac{1}{c}\hat{v} \times \hat{s}, \tag{12}$$

(4) $$m\hat{v} + \frac{1}{c^2}\hat{a} \times \hat{s} = o, \tag{13}$$

(4) $$m = \frac{1}{c^4}\hat{a} \cdot \hat{v} \times \hat{s} = \frac{1}{c^3}\hat{a} \cdot \hat{q}, \tag{14}$$

(5) $$m = \dot{M} - \frac{1}{c^2}\hat{G} \cdot \hat{v} = \frac{M_c\left(1 - \frac{1}{c^2}\hat{V} \cdot \hat{v}\right)}{\sqrt{1 - V^2/c^2}}. \tag{15}$$

Inserting (12) in (11) we obtain

$$\hat{G} = M\hat{v} + \frac{1}{c^2}\hat{a} \times \hat{s} + \frac{1}{c^2}\hat{v} \times \hat{s}', \tag{16}$$

where $\hat{a} = \hat{v}'$ is the three-dimensional acceleration. M is given by III (35) as

$$M = \frac{M_c}{\sqrt{1 - V^2/c^2}}, \tag{17}$$

and $\hat{V}$ is defined by III (32) as equal to $\hat{G}/M$.

Equation (16) expresses the aforesaid relation between the velocity and the momentum, cf. reference (3).

Calculating $\hat{G} \times \hat{v}$ from (16), and comparing the result with (10), we get

$$\hat{v} \cdot \hat{s} = o, \tag{18}$$

as $\hat{v} \cdot \hat{s}' = o$, because of (10), and $\hat{a} \cdot \hat{v} = o$, by differentiating the relation $\hat{v} \cdot v = c^2$.

Since $\hat{v}$ remains here always perpendicular to $\hat{s}$ (not only in the rest-system of the circle as in the first case) we have from (12) $q = s$, and therefore

$$\frac{1}{2} s_{\alpha\beta} s^{\alpha\beta} = s^2 - q^2 = o, \tag{19}$$

a result already obtained by a different method in II, Section 8.

Differentiating (18) with respect to t, we have

$$\hat{a} \cdot \hat{s} = o, \tag{20}$$

as $\hat{v} \cdot \hat{s}' = o$ by (10). Thus the spin vector remains perpendicular not only to the velocity but also to the acceleration.

From (13) we can now obtain the formula

$$\hat{a} = \frac{mc^2}{s^2} \hat{v} \times \hat{s}. \tag{21}$$

Comparing this with (12) we see that the vectors $\hat{q}$ and $\hat{a}$ are parallel and have the same direction (when $m > o$).

With the help of (18) and (20) we can deduce the following formulae, supplementary in a certain sense to (12) and (21),

$$\hat{v} = \frac{c}{s^2} \hat{s} \times \hat{q} = \frac{1}{mc^2} \hat{s} \times \hat{a}, \tag{22}$$

$$\hat{s} = \frac{1}{c} \hat{q} \times \hat{v} = \frac{s^2}{mc^4} \hat{a} \times \hat{v}. \tag{23}$$

Finally, we may notice that in the case of a free particle $\hat{G} \cdot \hat{s}' = o$, owing to (10), and hence, since $\hat{G}$ is a constant,

$$\hat{G} \cdot \hat{s} = \text{const.} \tag{24}$$

More generally, (24) holds good for every plane motion since in that case

$$\hat{s}' \times \hat{s} = (\hat{G} \cdot \hat{s}) \hat{v}, \tag{25}$$

as can be seen from (10) and (18), and since by (18) and (20) $\hat{s}$ has than a constant direction perpendicular to the plane of the motion,

the left-hand member of (25) vanishes, and so does $\hat{G} . \hat{s}$ (as v is different from zero).

6. In the proper system of the circle, Σ_c (which has been defined as an inertial system in which $\hat{G} = o$), $m = M_c$ by (15) and $\hat{s} = \text{const.}$ by (10). Equation (13) becomes then identical with equation II (25), by means of which the general solution of the equations of motion of a free spin-particle moving with the velocity of light were found in II. In Σ_c such a particle travels uniformly along a stationary circle in a plane perpendicular to $\hat{s}$, the sense of the motion being such that the moment of velocity with respect to the center of the circle is antiparallel to $\hat{s}$. If we imagine instead of the point singularity an infinitely small rotating sphere producing by its rotation the angular momentum $\hat{s}$, the rotation of the sphere about its axis and its motion along the circle will take place in opposite directions.

In Mathisson's method (as in every similar method of deducing the equations of motion of a point singularity from the differential equations of the gravitational field) a world tube is constructed cutting out a certain neighbourhood of the world-line of the singularity, outside which the field is to be at any rate regular. In our case all depends on whether the transversal dimensions of that tube are small or large in comparison with r_c, the proper radius of the circle. In the first instance, which should correspond to what Mathisson has worked out, this tube in four-dimensional space takes the form of a srew-line and does not tend to a geodetic line when its transverse dimensions decrease indefinitely and approach zero as a limit. Whereas in the second instance the tube, containing now the whole circle, may run in general much more smoothly and have approximately the shape of a geodetic line.

7. As previously noticed, the transition from the first case to the second one cannot be performed by merely accelerating the spin-particle up to the velocity of light, the rest-mass m_0 and the rest-spin s_0 remaining constant. However, we may consider that transition from a purely mathematical point of view as a transition from $v < c$ to $v = c$ in such a way that all the components of G^α and of $s^{\alpha\beta}$ should remain finite; for example, we may set in an arbitrary chosen frame of reference: $\hat{G} = o, \hat{s} = \text{const.}$ and $M \equiv G^4/c = \text{const.}$

It seems worth mentioning that the rest-mass m_0 of the spin-particle tends hereby to infinity, and not to zero, as is the case with photons. The rest-mass of a photon is always taken to be zero, as the energy of a particle with finite rest-mass (and without spin) moving with the velocity of light would be infinite. Here, however, the ex-

pression for the energy of a spin-particle moving with a velocity $v < c$ consists of two terms (cf. III (44)):

$$W = \gamma \left(m_0 c^2 - \frac{\gamma}{c} \hat{a} \cdot \hat{q} \right), \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{26}$$

and since for v tending to c the second term in parentheses [4] tends to + ∞ so must the first one too in order that W may remain finite. This can be seen also from the relation III (36) between M_c and m_0.

On the other hand, s_0 approaches zero as a limit when v tends to c; in a special case this may be inferred from III (49), and generally by comparison of III (50) with (19) of this paper.

8. Now we shall approach the question of the constants of motion of our system of equations. We have to distinguish between two kinds of such quantities; both are constant along the world line of the particle in virtue of the equations of motion, but while the quantities of the first kind may acquire arbitrary constant values, those of the second kind are always equal to zero (or to another fixed number).

In the first case there are two constants of motion of the first kind, namely m_0 and s_0. Neither of these quantities exists in the second case, and $\mathfrak{m}$, which superseded m_0, is a «scalar depending on the parametrization» and as such it could only be maid constant by a special choice of the parameter (see Section 10 below), but this constancy could not have any direct physical meaning.

In the next Section $M_c\, s_c$ will prove to be constant. So far as I can see it is the only constant of the first kind in the second case. M_c and s_c by themselves are only approximately constant under certain conditions which will be discussed in the next paper.

To the second kind belongs

$$w_\alpha w^\alpha = 0 \tag{27}$$

if we take (1)—(4) as the fundamental equations of our dynamics (and not (1)—(3) together with (27) from which (4) has been derived in Section 1). To establish this result we multiply (4) by w_α and taking (3) into account obtain $\mathfrak{m} w_\alpha w^\alpha = 0$; since $\mathfrak{m}$ is different from zero (cf. II, reference ([3])) (27) follows.

The bivector $s^{\alpha\beta}$ satisfies the following relations:

$$s^{\alpha\beta} w_\beta = 0 \quad (28.1) \qquad s^{[\alpha\beta} w^{\gamma]} = 0 \quad (28.2)$$

$$s_{\alpha\beta}\, s^{\alpha\beta} = 0 \quad (28.3) \qquad s^{[\alpha\beta} s^{\gamma\delta]} = 0. \quad (28.4)$$

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[4] In the limit $\hat{a}$ and $\hat{q}$ tend to become parallel and different from zero, as can be seen from (12) and (21).

The first of these was assumed at the very beginning as the condition of non-existence of negative mass. The fourth follows immediately therefrom, since the determinant of the four homogeneous equations (28. 1), which obviously must vanish, is equal to the square of the expression on the left-hand side of (28. 4). Equation (28,2) may be deduced from (28. 4) and (4) as follows:

$$m\, s^{[\alpha\beta} w^{\gamma]} = -\frac{1}{c^2}\, s^{[\alpha\beta} \dot{s}^{\gamma]\sigma} w'_\sigma = o \tag{29}$$

as it is easy to verify that owing to the antisymmetry of $s^{\alpha\beta}$

$$s^{[\alpha\beta} \dot{s}^{\gamma]\sigma} = s^{[\alpha\beta} \dot{s}^{\gamma\sigma]}.$$

Finally, the relation (28. 3), which has been proved already by other methods (cf. (19) and II (33)), is also an immediate consequence of (28. 1), (28. 2) and (3), as

$$s_{\alpha\beta}\, s^{\alpha\beta} w^\gamma = s_{\alpha\beta}\,(s^{\alpha\beta} w^\gamma + s^{\beta\gamma} w^\alpha + s^{\gamma\alpha} w^\beta) = 3\, s_{\alpha\beta}\, s^{[\alpha\beta} w^{\gamma]} = o. \tag{30}$$

The geometrical interpretation in four-dimensional Minkowskian space of the relations (28) is as follows. (28. 4) expresses the fact that $s^{\alpha\beta}$ is a flat, or one-sheet, bivector, i. e., that it may be written as an alternating product of two (perpendicular) 4-vectors. (28. 1) is the condition that w^α should be perpendicular to the plane of $s^{\alpha\beta}$, and (28. 2) that it should lie in that plane, both these conditions being obviously compatible for an isotropic vector only. Finally, (28. 3) expresses the fact that $s^{\alpha\beta}$ is isotropic, that is, that it lies in a tangent plane to the absolute cone.

The physical significance of the first two relations (28) may be also worth mentioning. If the particle possess a magneto-electric moment $\mu^{\alpha\beta}$ proportional to $s^{\alpha\beta}$, we may write instead of (28. 1) and (28. 2)

$$\mu^{\alpha\beta} w_\beta = o, \qquad \mu^{[\alpha\beta} w^{\gamma]} = o. \tag{31}$$

The same relations with the four-dimensional velocity u^α in place of w^α were interpreted in III, Section 3 as giving the condition that $\mu^{\alpha\beta}$ should be respectively «purely magnetic» and «purely electric», that is, that in the rest-system of the particle the electric and the magnetic moments respectively should vanish; incidentally, that would imply the vanishing of $\mu^{\alpha\beta}$. Here no rest-system of the particle exists, as the particle travels with the velocity of light, and both conditions (31) may be satisfied simultaneously. Moreover, (27) is an immediate consequence of the relations (31), as can be readily seen by multiplying the second one by w^α and allowing for the first one.

9. In the first case, the rest moment s_0 was a constant, as a result of $s_{\alpha\beta} s^{\alpha\beta}$ being a non-vanishing constant of motion. Here $s_{\alpha\beta} s^{\alpha\beta}$ is also a constant of motion, which may be seen from (3) and the equation resulting from (2) on multiplication by $s_{\alpha\beta}$, but moreover this constant is zero, and hence the only conclusion we can draw from (19) is that $q = s$ in every inertial frame of reference.

However, there exists a certain invariant value of s, namely «s in the inertial frame of reference in which $\hat{G}$ vanishes», that is s_c. The question arises if s_c is a constant of motion. We are now going to prove that

$$M_c s_c = \text{const.} \tag{32}$$

and thus the answer to the above question depends on whether M_c is a constant of motion or not. In the next paper M_c will be shown to be only approximately constant.

First, we shall prove that the magnitude of the 4-vector $s_{\alpha\beta} G^\beta$ is constant, i. e. that

$$\left|s_{\alpha\beta} G^\beta\right| = \text{const.} \tag{33}$$

In fact, half the derivative with respect to p of the square of that magnitude is (for $c = 1$)

$$s_{\alpha\beta} G^\beta (s'^{\alpha\gamma} G_\gamma + s^{\alpha\gamma} G'_\gamma) = s_{\alpha\beta} G^\beta [(G^\alpha w^\gamma - G^\gamma w^\alpha) G_\gamma + \varepsilon s^{\alpha\gamma} F_{\gamma\sigma} w^\sigma =$$
$$= \varepsilon F_{\gamma\sigma} s_{\alpha\beta} w^\sigma s^{\alpha\gamma} . G^\beta = 0. \tag{34}$$

The second equality follows from (3) and the antisymmetry of $s^{\alpha\beta}$, the third is a consequence of (28. 2) and (3) since

$$F_{\gamma\sigma} s_{\alpha\beta} w^\sigma s^{\alpha\gamma} = F_{\gamma\sigma} s_{\alpha\beta} w^\gamma s^{\sigma\alpha} = \frac{1}{2} F_{\gamma\sigma} s_{\alpha\beta} (w^\sigma s^{\alpha\gamma} + w^\gamma s^{\sigma\alpha}) =$$
$$= -\frac{1}{2} F_{\gamma\sigma} s_{\alpha\beta} w^\alpha s^{\gamma\sigma} = 0. \tag{35}$$

Thus (33) is established. To prove (32) it is now sufficient to remark that the magnitude of the 4-vector $s_{\alpha\beta} G^\beta$ is equal to $M_c s_c$ (since in the proper system of the circle $G^k = 0$, $s_{\alpha\beta} G^\beta$ reduces to one term $s_{\alpha 4} G^4 = \{\hat{q} M_c | 0\}$ and $q = s$ by (19).

Alternatively, the theorem (32) might be established by proving in a similar way that

$$\left|s^{[\alpha\beta} G^{\gamma]}\right| = \text{const.}, \tag{36}$$

the magnitude of the above trivector being also equal to $M_c s_c$.

10. *The parameter* π. The quantity m given by (5) being a «scalar depending on the parametrization», its variability along the world-

line of the particle also depends on that parametrization. By a suitable choice of the parameter, m may be made constant, the value of that constant being, of course, quite arbitrary; we shall denote it by n. Thus, putting $c = 1$, we define the new parameter π by the following equation:

$$- G_\sigma \frac{dx^\sigma}{d\pi} = n \tag{37}$$

or

$$d\pi = - \frac{G_\sigma dx^\sigma}{n}. \tag{38}$$

Obviously π is thus defined to within an arbitrary linear transformation. Alternatively, we can write instead of (38)

$$\frac{d\pi}{dp} = - \frac{G_\sigma w^\sigma}{n}, \tag{39}$$

and for $p = t$

$$\frac{d\pi}{dt} = - \frac{G_\sigma v^\sigma}{n}. \tag{40}$$

For a free particle in the rest system of the circle we have $G_k = 0$, $G^4 \equiv M_c = \text{const.}$, and hence $d\pi/dt = M_c/n$ or

$$\pi = \frac{M_c}{n} t. \tag{41}$$

We see, that the special parameter π is intimately related to the «proper time of the circle», that is the time coordinate t in the rest-system of the circle, but it cannot be identified with that proper time. So long as no external forces act on the particle, both these quantities are proportional, and they could even be made equal at the outset by a suitable choice of the constant n. But this equality would hold only so long as the particle is not acted upon by an electromagnetic field. Let us imagine that the particle enters such a field, and after remaining there for a while, gets out of it again into a field-free space; obviously, during the whole process n remains constant, as it is so by hypothesis, but M_c undergoes changes in the electromagnetic field, and, in general, after the passage of the particle through the «potential barrier» M_c will have another value than before. Thus, even if initially π had been chosen equal to the proper time of the circle, it would retain that property only so long as M_c might be looked upon as a constant. Consequently, we are again led to the question of the approximate constancy of M_c, which has been already postponed to the next paper.

11. Because of (28. 4) the bivector $s^{\alpha\beta}$ is flat, and may be therefore put in the form of an alternating product of two orthogonal 4-vectors. Let us write therefore

$$s^{\alpha\beta} = a^\alpha b^\beta - a^\beta b^\alpha, \quad a_\alpha b^\alpha = o, \tag{42}$$

whence

$$\frac{1}{2} s_{\alpha\beta} s^{\alpha\beta} = a_\alpha a^\alpha . b_\beta b^\beta, \tag{43}$$

and as by (28. 3) the bivector $s^{\alpha\beta}$ is isotropic the above expressions vanish and one of the two vectors a^α and b^β must be isotropic. Both could not be, since they are orthogonal, and two orthogonal and isotropic 4-vectors in Minkowskian space are easily proved to be parallel, which cannot be the case here as $s^{\alpha\beta}$ would then vanish. So let b^α be the isotropic vector, then, thanks to (28. 1) and the theorem just stated, b is parallel to w^α, and we may write instead of (42) (after multiplication of a^α by a suitable factor)

$$s^{\alpha\beta} = a^\alpha w^\beta - a^\beta w^\alpha. \tag{44}$$

Thus all the four relations (28) are satisfied. Yet we may go still a step further and identify a^α with a multiple of w'^α. To this end, let us differentiate (44) with respect to p, and compare the result with (2). We have

$$X^\alpha w^\beta - X^\beta w^\alpha = a^\alpha w'^\beta - a^\beta w'^\alpha, \tag{45}$$

where

$$X^\alpha = G^\alpha - a'^\alpha,$$

and hence w^α lies in the plane of the 4-vectors a^α and w'^α. We may write therefore

$$a^\alpha = \mathfrak{A} w'^\alpha + \mathfrak{B} w^\alpha. \tag{46}$$

Inserting in (44), we get finally

$$s^{\alpha\beta} = \mathfrak{A}(w'^\alpha w^\beta - w'^\beta w^\alpha). \tag{47}$$

Here $\mathfrak{A}$ is a «scalar depending on the parametrization».

12. In this last Section we shall find an expression for $\mathfrak{A}$ together with some further interesting formulae.

Differentiating (5) with respect to p and bearing in mind that, in virtue of (6),

$$G'_\alpha w^\alpha = o, \tag{48}$$

we have

$$m' = -\frac{1}{c^2} G_\alpha w'^\alpha \tag{49}$$

Multiplying (47) by w'^{β} and inserting in (4) we have

$$\mathfrak{m} w^{\alpha} = -\frac{1}{c^2} s^{\alpha\beta} w'_{\beta} = \frac{\mathfrak{A}}{c^2} w'_{\beta} w'^{\beta} . w^{\alpha}. \tag{50}$$

Hence

$$\mathfrak{A} = \frac{\mathfrak{m} c^2}{w'_{\beta} w'^{\beta}}. \tag{51}$$

Multiplying (47) by G_{β}, we have, thanks to (5) and (49),

$$s^{\alpha\beta} G_{\beta} = -\mathfrak{A} c^2 (\mathfrak{m} w'^{\alpha} + \mathfrak{m}' w^{\alpha}) \tag{52}$$

and so

$$\frac{1}{c^2} \left| s^{\alpha\beta} G_{\beta} \right|^2 = M_c{}^2 s_c{}^2 = \mathfrak{A}^2 c^2 \mathfrak{m}^2 w'_{\alpha} w'^{\alpha}. \tag{53}$$

Combining (51) and (53) we have

$$\mathfrak{A} = \frac{M_c{}^2 s_c{}^2}{\mathfrak{m}^3 c^4} \tag{54}$$

and

$$w'_{\alpha} w'^{\alpha} = \frac{\mathfrak{m}^4 c^6}{M_c{}^2 s_c{}^2}. \tag{55}$$

Substituting in (47) from (54) we get the interesting formula

$$s^{\alpha\beta} = \frac{M_c{}^2 s_c{}^2}{\mathfrak{m}^3 c^4} (w'^{\alpha} w^{\beta} - w'^{\beta} w^{\alpha}). \tag{56}$$

Incidentally, it may be noticed that thanks to (55) a third proof of the constancy of $M_c s_c$ may be reached as follows. Multiplying (2) by w'_{β}, we get, thanks to (49),

$$s'^{\alpha\beta} w'_{\beta} = \mathfrak{m}' c^2 w^{\alpha}. \tag{57}$$

Bearing this in mind, and differentiating (4) with respect to p, we obtain

$$2 \mathfrak{m}' w^{\alpha} + \mathfrak{m} w'^{\alpha} + s^{\alpha\beta} w''_{\beta} = 0. \tag{58}$$

Since

$$w''_{\alpha} w^{\alpha} = -w'_{\alpha} w'^{\alpha} \tag{59}$$

as a conseequence of (27), and

$$2 w''_{\alpha} w'^{\alpha} = (w'_{\alpha} w'^{\alpha})', \tag{60}$$

equation (58) multiplied by w''_{α} goes over in the following differential equation for $w'_{\alpha} w'^{\alpha}$

$$-4 \mathfrak{m}' . w'_{\alpha} w'^{\alpha} + \mathfrak{m} (w'_{\alpha} w'^{\alpha})' = 0. \tag{61}$$

the solution of which is

$$\frac{w'_\alpha w'^\alpha}{\mathfrak{m}^4} = \text{const.} \tag{62}$$

The comparison of this result with equation (55) proves once more the constancy of $M_c s_c$.

All the above formulae and calculations may be simplified by introducing the special parameter π defined in Section 10. In that case

$$\mathfrak{m} = \mathfrak{n} = \text{const}, \quad \mathfrak{m}' = 0 \tag{63}$$

and $\mathfrak{A}$, the «scalar depending on the parametrization» becomes a constant.

ON TWO RELATIVISTIC MODELS OF DIRAC'S ELECTRON*

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1. The view has been often expressed that some at least of the difficulties of the present quantum theory of fields arise from the inadequatness of the underlying «classical model», rather than from the inadequatness of the methods of quantization. Possibly the same may be true of Dirac's theory of the spinning electron. The difficulties there encountered are probably more a matter of interpretation than of the equation used being defective. Perhaps some progress might be made by substituting a spin-particle obeying the laws of relativistic dynamics to a spinless particle, used as starting point by Dirac. Of course, the quantization would then have to be performed in another way, as (a), the equations of motion are reducible to a differential equation of the third order (or at least intimately connected with such an equation), and cannot be brought to the canonical Hamiltonian form (at any rate if no auxiliary variables are introduced), and (b), the spin of the particle has been introduced beforehand, previously to the quantization, and does not appear only as an eventual byproduct of that quantization.

We shall see presently that there exist at least two different relativistic models of a material particle with spin, corresponding to the first and second cases dealt with in my previous papers. The *second model* seems to be by far the more interesting, but, as the *first* present also some advantages of its own, both models will be discussed here. However, before entering upon the subject proper it may be well to insert some remarks concerning the radiation of a moving point-charge carrying a magnetic moment.

2. According to classical electrodynamics an electric point-charge ε, where ε is the charge of an electron, travelling with the enormous frequency of Schrödinger's *Zitterbewegung* along a circle the diameter of which would be of the order of h/m_0c would radiate with a very high intensity. It is true that a classical model cannot be expected to give a fair account of the radiation emitted

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* See I, reference (*). The four preceding papers will be designated henceforth by I, II, III and IV.

by an electron, but such large radiation without counterpart in nature would render any correspondence between classical and quantum theoretical behaviour of an electron impossible. I remarked as long ago as 1938 *(1)* that the situation changes radically if one takes into account not only the charge of an electron but also its magnetic moment. It can than happen that, under suitable conditions, the radiation due to the revolving magnetic dipole cancels, approximately at least, the radiation emitted by the revolving point-charge. The argument presented in 1938 was carried through in non-relastivistic approximation only, but it can be applied also in the general case, if we interpret all the quantities involved as referring to the «rest-system of the circle». This improvement is necessary to draw the conclusions aimed at in the present Section.

For the convenience of the reader I repeat here the argument of 1938 with only slight modifications. We know that a magnetic dipole with a moment $\hat{\mu}$ moving with the velocity $\hat{v}$ produces an electric moment

$$\hat{\pi} = \frac{1}{c} \hat{v} \times \hat{\mu}. \tag{1}$$

Suppose such a magnetic dipole carrying an electric charge ε to be in uniform motion around a circular orbit of radius r, the angular velocity being ω and the dipole axis remaining perpendicular to the plane of the circle, the electric moment (1) will then be directed along the radius vector of the particle. If we put

$$\pi = \varepsilon r \tag{2}$$

and assume that the sense of motion is such as to produce a magnetic moment opposite to $\hat{\mu}$ (and hence an electric moment pointing inwards for $\varepsilon > o$ and outwards in the opposite case), the action of the resulting electric moment will be equivalent, at sufficiently distant points, to that of a charge $+\varepsilon$ in the centre of the circle and a charge $-\varepsilon$ on the particle. The action of the moving electric charge will be thus compensated, and there will remain only a charge $+\varepsilon$ at rest in the centre of the circle. Hence, at large distances, there will remain only an electrostatic field and the radiation will disappear. Putting $v = \omega r$ and $\omega = \frac{Mc^2}{s}$ we get, from (1) and (2), for the ratio of the magnetic moment to the angular momentum of the particle

$$\varkappa = \frac{\mu}{s} = \frac{\varepsilon}{Mc}. \tag{3}$$

Hitherto all the formulae were valid for both cases alike, as $\omega = Mc^2/s$ in both cases if all symbols are understood to represent quantities in the rest-system of the circle. From now on we must discern the two possibilities.

In the first case

$$M = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \tag{4}$$

m_0 is a constant representing the rest-mass of the electron, and as for an electron $\varkappa = \varepsilon / m_0 c$ we see that the condition of vanishing radiation is only satisfied for infinitesimal velocities (on the circle in its rest-system), that is, for an infinitesimal proper radius r_c.

In the second case, as we shall see presently, M itself has to be put equal to the mass of an electron and (3) is always satisfied.

3. The first model mentioned above is very much the same as the one considered as far back as 1926 by Thomas and Frenkel. Our results in I and III together with the contents of the preceding Section suggest however a slightly different interpretation of its behaviour. The equations of motion considered hitherto did not take into account the reaction force of the radiation; according to them the motion of the particle consisted, broadly speaking, of a motion along a geodetic line together with a superimposed circular motion of proper radius r_c. As these equations are equivalent to a differential equation of the third order, the general solution is far more complicated than in the case of a particle without spin. But, the reaction of the radiation, instead of still further complicating the motion, simplifies it materially, as it does not allow the circular motion to develop, an the spin-particle is again moving along a geodetic line or, at least, infinitely near to it. We can also express this fact by saying that the proper radius of the circle is infinitely small.

The chief advantage of this first model over the second one consists in the fact that — as was shown in detail by Frenkel — it gives a fair account of the behaviour of a spinning electron in a magnetic field (in so far as we really know how such an electron should behave classically).

Besides that one superiority the first model does not possess any other over the second one; in particular, it does not show so many striking analogies with the behaviour of Dirac's electron. Before passing to the consideration of the latter, one more remark may be added. There exists a close analogy between both models; though in the second case the proper radius of the circle has a finite value $h/2M_0c$, it is obvious that this radius must be considered as «unobser-

vable», or, in other words, that the consequence drawn from the second model have a physical meaning only in so far as this radius may be considered as infinitely small.

4. The second model consists of a point singularity moving with the velocity of light according to the laws exposed in II and IV. The essential point however is that it is not the singulariy itself, but rather its mean position — or the small circle on which it moves in an appropriately chosen frame of reference — which has to be considered as representing the electron.

We say «electron» as we shall discuss here especially the analogies of our model with D i r a c's theory of the electron, but it seems probable that all elementary particles endowed with spin [1] must have the same classical model and it is only by the process of quantization that the individuality of different sorts of particles is brought in.

5. We shall now enumerate and discuss some analogies and some promising differences between our relativistic model of an electron and its quantum-mechanical counterpart.

(a) The momentum $\hat{G}$ ceases to be parallel to $\hat{v}$ just as in the theory of D i r a c where the momentum and the velocity have different operators.

(b) The magnitude of the «kinematical velocity» $\hat{v}$ is always c; the values of its components in any direction range between —c and +c. In D i r a c's theory the eigenvalues of the operators of the velocity components $c\alpha_x$, $c\alpha_y$, $c\alpha_z$ are $\pm$ c. The correspondence between the classical and the quantum theoretical behaviour is just the same as with the spin vector and its components.

Notice that we were led, in IV, to introduce also another concept of velocity, called «dynamical velocity». S c h r ö d i n g e r *(2, 1930)* has done almost the same in the theory of the Dirac electron; he calls that new velocity «macrovelocity», in contradistinction to v, the «microvelocity».

(c) In both theories the moments of the linear momentum are not constants of motion by themselves but they may be supplemented in such a way as to acquire that property; the additional terms involve the vector of angular momentum, or its operator, which are thus introduced in a very similar manner.

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[1] If the singularities representing them do not contain in addition to gravitational unipoles and dipoles also gravitational quadrupoles, octupoles, etc. It will be shown in a subsequent paper that the mean position of a quadrupole singularity moves also along a geodetic line.

It could rightly be objected that the above analogies are not convincing, as our model was expressly constructed to bring them about. This is not altogether true, as it seems already interesting enough that this could have been done in so natural a way. At any rate the following analogies are free from those objections, as they follow automatically from the initial assumptions.

6. (d) If we put M_c equal to the mass of an electron and s_c equal to h/2, as it is the case with an electron, then the frequency of the rotation around the circle (in the rest-system of the circle) will be just equal to the frequency of S c h r ö d i n g e r s *Zitterbewegung*.

(e) Under the above conditions the proper diameter of the circle

$$2r_c = \frac{h}{M_c c} \tag{5}$$

becomes equal to the well-known «wave-length of the Compton shift». It is the wave length which would be produced if the energy of an electron were transformed entirely into a photon; moreover, it is the maximum accuracy with which the position of a particle of mass M_c may be ascertained. This last result fits nicely into our theory as we are bound to admit that the motion of the singularity around the circle is unobservable and that it is only the circle as a whole which plays the role of an electron. The amplitude of the *Zitterbewegung* has the same order of magnitude as the radius of the circle in our case.

(f) Two isotropic tensors, w^α, for which $w_\alpha w^\alpha = o$, and $s^{\alpha\beta}$, for which $s_{\alpha\beta} s^{\alpha\beta} = o$, play a dominant role in our theory. It may be interesting to remark that isotropic four-vectors as well as isotropic four-dimensional bivectors are closely connected with spinors.

7. (g) The motion of the spin-singularity in an electromagnetic field is far more complicated than in a field-free space, but as a first approximation, if the intensity of the field is small enough, we may go on speaking of a motion on a circle even in an eléctro-magnetic field, the circle being subject to a small acceleration as a whole and to slight deformations. Obviously, the inequality to be fulfilled by the intensity of the field may be obtained by expressing the fact that in the rest-system of the circle the displacement of the center of the circle during the time of one revolution

$$T = \frac{2\pi}{\omega} = \frac{\pi h}{M_c c^2} \tag{6}$$

should be vanishingly small in comparison with the proper radius of the circle

$$r_c = \frac{2\,M_c c}{h}. \tag{7}$$

It is easy to prove that the acceleration of the center of the circle in not too strong electromagnetic fields is the same as for an electric point-charge without spin. Hence, we can write

$$\frac{1}{2}\,aT^2 = \frac{1}{2}\,\frac{\varepsilon E}{M_c}\,T^2 \ll \frac{h}{2\,M_c c}, \tag{8}$$

or, omitting the irrelevant factor π^2,

$$\frac{\varepsilon}{M_c c^2}\,Eh \ll M_c c. \tag{9}$$

Curiously enough it is just the same inequality which presents itself in Dirac's theory of the electron as the condition of no jumps from positive to negative energy states. It is found there by expressing the fact that the change of the potential energy of an electron over distances of the order of magnitude $h/M_c c$ should be very small in comparison with $M_c c^2$, i. e.

$$\varepsilon E \cdot \frac{h}{M_c c} \ll M_c c^2. \tag{10}$$

This inequality is equivalent to (9).[2]

It may be also written in an invariant form, as follows:

$$\frac{\varepsilon}{M_c c^2}\,s^{\alpha\beta} F_{\alpha\beta} \ll M_c c^2. \tag{11}$$

8. (h) *Variability of mass and pair production.* Strictly speaking all what has been said hitherto lacks real foundation so long as we have not yet proved that M_c is a constant of motion and that it may therefore be put equal to the mass of an electron.

In IV it was only proved that

$$M_c s_c = \text{const.}, \tag{12}$$

---

[2] Cf. L. de Broglie, *L'électron magnétique*, Paris 1934, p. 289. De Broglie writes: «Ce résultat conduit á penser que s'il on pouvait s'interdire de considérer des distances spatiales inférieures à h/m_0c, on parviendrait peut-être à éliminer les ondes à énergie négatives».

and it can be easily shown that M_c and s_c separately are not constant. At first sight this state of affairs could seem a serious drawback of the present theory, but at closer inspection it turns out to be one of its most attractive features, as we shall prove immediately that M_c is approximately constant if the field through which the electron passes do not change to abruptly, and that the variability of mass is intimately connected with the production of pairs.

Let us put the question in the following way. At the outset, let the electron move in a field-free space; it is than represented by a circle of constant proper radius r_c, constant mass M_c, and, thanks to (12), constant magnitude of angular momentum s_c — all these quantities measured in the rest-system of the circle. They are linked together by the equation

$$r_c = \frac{s_c}{M_c c}. \tag{13}$$

Now, let the electron enter an electromagnetic field and after remaining there for a while get out of it into a field-free space again. There, it becomes once more a regular circle (the rest-system of which is in general different from what it has been before the passage through the field). But do M_c and hence r_c and s_c return to their previous values? Strictly speaking the answer to this question must be in the negative, but we shall prove now that M_c is approximately constant when the field satisfies the following condition

$$|\partial_\gamma F_{\alpha\beta}| \ll \frac{2 M_c c}{h} |F_{\mu\nu}| \tag{14}$$

in which s_c has been put equal to $h/2$. The proof runs as follows. Taking into account equation IV (6) we can write

$$\frac{d}{dp} G^\alpha G_\alpha = 2\, G^\alpha G'_\alpha = 2 \frac{\varepsilon}{c} G^\alpha F_{\alpha\sigma} w^\sigma = \frac{\varepsilon}{c} F_{\alpha\sigma}(G^\alpha w^\sigma - G^\sigma w^\alpha) = \frac{\varepsilon}{c} F_{\alpha\sigma} s'^{\alpha\sigma} =$$

$$= \frac{\varepsilon}{c} \frac{d}{dp}(F_{\alpha\sigma} s^{\alpha\sigma}) - \frac{\varepsilon}{c} s^{\alpha\sigma} F'_{\alpha\sigma}$$

and hence, as $-G^\alpha G_\alpha / c^2 = M_c^2$,

$$\frac{d}{dp}\left(M_c^2 + \frac{\varepsilon}{c^3} s^{\alpha\beta} F_{\alpha\beta}\right) = \frac{\varepsilon}{c^3} s^{\alpha\beta} F'_{\alpha\beta} = \frac{\varepsilon}{c^3} s^{\alpha\beta} w^\gamma \partial_\gamma F_{\alpha\beta}. \tag{15}$$

The second term in parentheses is small in comparison to the first one, in virtue of (11). To write down the condition for neglecting the right-hand member of (15), we must express the fact that this term multiplied by the change of p during one revolution around the circle is vanishingly small in comparison with $\varepsilon s^{\alpha\beta} F_{\alpha\beta}/c^3$. As the

equation does not depend on the choice of the arbitrary parameter p we may put $p = t$, and hence $w^\gamma = v^\gamma$. To be sure that the sum on the right hand side of (15) is small enough, we must require that all its members should be small enough. Thus, remembering (6), we obtain the inequality (14).

This condition has a simple meaning in the theory of Dirac, at least when the external electromagnetic field is a plane monochromatic wave of frequency ν. Then $\frac{\partial E}{\partial x}\Big/E = \nu/c$ and a special case of (15) reads

$$h\nu \ll M_c c^2. \tag{16}$$

Thus, we see that the condition of constancy of mass (14) is equivalent to the condition of non-production of pairs. This is again a very sensible result. So long as we are very far from the possibilities of pair production the mass of the electron is constant. Of course, we get no sharp limit, as classical theories never give sharp ones.

9. It is not at all clear how the quantization of our relativistic model has to be performed. In any case, it will have to be done in a very different manner from the present one, as Schrödinger's *Zitterbewegung* is a consequence of the superposition of states of positive and states of negative energies, and to day we do imagine that a particle is either in a state of positive *or* in a state of negative energy, whereas our circular model of an electron, being a model of the *Zitterbewegung*, must correspond simultaneously to states of positive and states of negative energies.

References

(1) J. W. Weyssenhoff, Nature **141**, 328 (1938).
(2) E. Schrödinger, Ber. d. Preuss. Ak. 418 (1930).

A COUNTER APPARATUS FOR THE MEASUREMENTS OF COSMIC RAYS

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(Received April 19, 1947)

For the measurements of the weak effects of cosmic rays, e. g. penetrating showers or the component of great depth, we must use G-M counters of large dimensions. They must stand long use, because the investigations last several months or more. We describe here the construction of counters of large dimensions, which can be made in any laboratory fairly simply, which are distinguished by mechanical solidity without being sealed in glass, and which have all the properties of good counters with respect to plateau, efficiency, stability during work, etc.

The mechanical construction of the counters.

The counter shown in fig. 1 is set up in a brass tube AA, which with the lid B and the collar C forms a cylindrical brass chamber used as a cathode. The central wire D, the anode, is fixed in a glass insulator E stretched from a small spring F through the centre of a glass tube G sealed into the collar C. Both these glass tubes enter the counter in order to prevent a discharge between the wire and the near walls of the chamber. On the end of the glass tube G outside the chamber is a brass cap H in the centre of which the wire is fixed. After setting up the complete counter, we seal a glass tube to the brass cap in order to join the counter with the vacuum- and filling-apparatus. This glass tube is sealed off afterwards.

To make the counter vacuum-tight all joints, brass-brass and brass-glass, are sealed with tin (25 p. c. Pb). The glass tubes prepared for sealing with brass were chemically deposited with a thin film of platin (from H_2PtCl_6) [1] then electrolitically coppered and sealed with tin.

[1] E. Angerer, Technische Kunstgriffe bei physikalischen Untersuchungen, Sammlung Vieweg.

Before setting up the counter, all its brass parts were cleaned first with slacked lime, then with technical nitric acid, and then rinsed with water. This procedure was used because of the very bad state of the brass tubes. It has been ascertained that small deformations of tubes, such as their imperfect circularity or not quite coaxial position

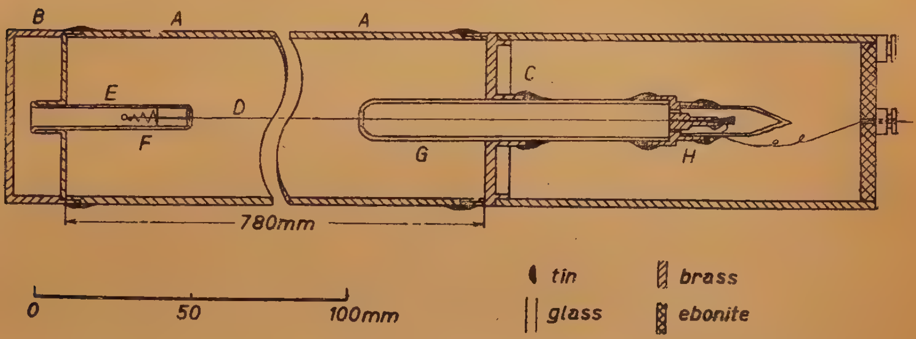

Fig. 1. Large counter for the measurements of cosmic rays.

of the wire, do not influence visibly the behaviour of the counter. The counters were thoroughly evacuated before filling. As anode we have used a resistance wire (diameter 0,1 mm). The dimensions of one of the counters are indicated in fig. 1.

Filling of the counters.

It is well known that one distinguishes counters filled with a gas or a mixture of gases and those filled with a gas mixed with an organic vapour. In the first type the discharge is quenched by applying a high resistance ($10^9\,\Omega$ or more) or a radio tube in a circuit of the Neher-Harper type *(1)*. In the other type, called «self-quenching», the recovery of the counter takes place in consequence of the discharge itself, without the essential influence of the high resistance *(2, 3)*. This second type of counters is mostly used now in consequence of its advantages over the first, and most frequently a mixture of argon and alcohol vapour is used.

We have investigated counters filled with pure vapour of aceton or alcohol to a pressure of 10 mm Hg. The counters with aceton vapour exhibited very good properties at first after filling. The threshold voltage of a counter of the dimensions shown in fig. 1 is about 1370 V, the length of plateau 400—600V, the efficiency $>$ 98,5%. If however they have been in action for a longer time they ceased to work satifactorily. With the resistance of $10^8\,\Omega$ they worked well for about

a month, counting in this time about 10^7 pulses. Therefore they cannot be adopted for measurements over a longer period of time.

The mixture mostly used in self-quenching counters is one of argon with alcohol-vapour. We have used this mixtures because now it may be considered as a normal mixture in counter-filling.[2]

The counter reproduced in fig. 1 filled with alcohol vapour (absolute alcohol) under a pressure of 10 mm Hg and argon under a pressure of 90 mm Hg shows the following properties:

The threshold voltage (indicated with an oscillograph)			1100 V.
The lowest voltage of correct counting			1130 V.
The length of plateau for resistance	$10^8\,\Omega$	about	400 V.
" " " " " "	$10^7\,\Omega$	"	350 V.
" " " " " "	$10^6\,\Omega$	"	250 V.

The plateau is meant as the voltage region in which the recorder gives constant count-rates and the pulses are singular. The number of counts without any radioactive substance is about 700/min. The above data refer to 10 counters filled simultaneously. The differences in the threshold voltages were not greater than 20 V. The length of the plateaus for all the ten counters was identical within our possibilities of estimation. As the counters filled with pure aceton vapour got spoiled after a certain time, we put the argon-alcohol counter under the influence of a small radioactive source giving in each counter about 2.10^4 pulses/min. during 485 hours at 200 V above the threshold voltage with a resistance of $10^8\,\Omega$. After the removal of the radioactive substance the threshold voltage of the irradiated counter got about 70 V higher and the length of the plateau shrinked to about 200 V. The counter was still good enough to be used.

Coincident circuits.

The resolving time for accidental coincidences of a conventional Rossi-circuit depends first of all on the following factors. (a) The grid resistance of the Rossi tube, (b) The grid voltage of this tube, (c) The voltage of the counter, (d) The grid voltage of the tube recording the coincidences (thyratron). This is the consequence of the following circumstances: (a) The form of the pulse on the common anode depends on the time constant of the system connected with the grid of the Rossi tube. If the capacity of this system remains unchanged, this time-constant will depend only on the grid resistance.

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[2] We are greatly indebted to Prof. H. Niewodniczański and Prof. J. Weyssenhoff for supply of argon.

(b) The increase of the positive voltage within certain limits does not change the size of the pulse (in volts) on the anode, but narrows it in the upper part, according to the work-characteristic of the tube. (c) As the resolving time depends on the size of the pulse (in volts) the counter voltage influences it. (d) The pulses deriving from the accidental coincidences are not always equal, since their size

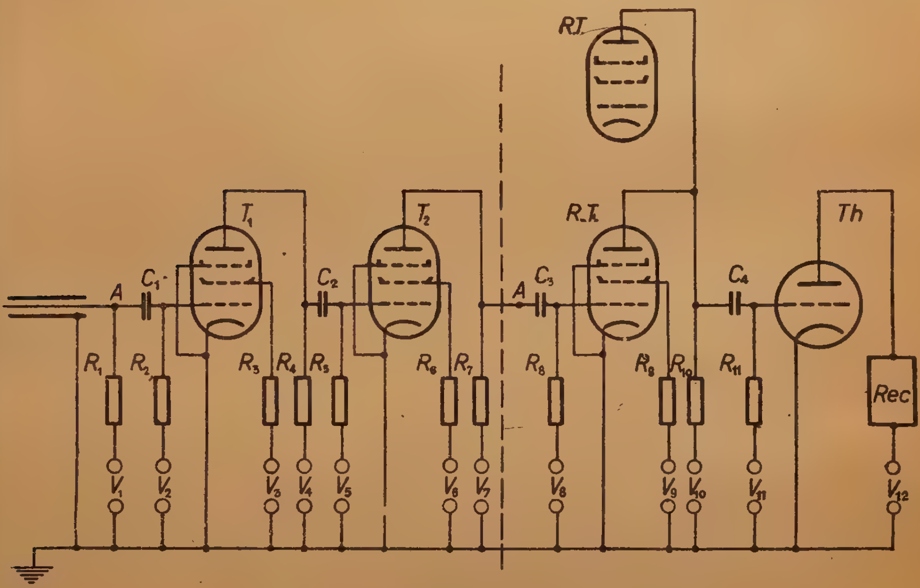

Fig. 2.

Data for simple Rossi coincident circuit with resolving time 8.10^{-6} sec. Counter connected to point A': $R_1 = 10^7\ \Omega$, $R_8 = 20\ k\Omega$, $R_9 = R_{10} = 250\ k\Omega$, $R_{11} = 1\ M\Omega$, $C_3 = 50$ pF, $C_4 = 500$ pF, $V_1 = +1300$ V, $V_8 = +4$ V, $V_9 = +115$ V, $V_{10} = +230$ V $V_{11} = -23$ V; $V_{12} = +150$ V.

Data for high-resolving power coincident circuit with a resolving time 1.10^{-6} sec. Counter connected to point A: $R_1 = 10^7\ \Omega$, $R_2 = R_4 = R_5 = R_7 = 50\ k\Omega$, $R_3 = R_6 = R_9 = R_{10} = 250\ k\Omega$, $R_8 = 10\ k\Omega$, $R_{11} = 1\ M\Omega$, $C_1 = C_2 = C_3 = 50$ pF, $C_4 = 500$ pF, $V_1 = +1300$ V, $V_2 = +5$ V, $V_5 = -7$ V, $V_8 = +5$ V, $V_{11} = -27$ V, $V_8 = V_6 = V_9 = +115$ V, $V_4 = V_7 = V_{10} = +230$ V, $V_{12} = +150$ V, $T_1 = T_2 = R.T. = AF\,7$, Th = AC 50, Rec. Recorder.

depends on the time interval between the pulses coming from both counters. These pulses are therefore generally smaller than the pulses of the real coincidences. In consequence we can bias the grid of the thyratron with negative voltage so that we may omit the smaller pulses. In relation to (c) we must still remark that the plateau of a single counter in a given circuit begins with a somewhat lower voltage than the plateau obtained for coincidences. The working voltage for counters, therefore, must not be too high but within the limits of the

plateau of the real coincidences. Taking these circumstances into account, we can, using the common Rossi circuit, attain the resolving time of 1.10^{-5} sec. without great difficulties. The right part of fig. 2 separated from the rest by a dotted line shows such a simple circuit. It can be used as it is only when the pulses arriving to the grids of the Rossi tubes are equal, as for single counters connected with the grids; thus, if we want to use this circuit for sets of counters, we must equalize the pulses before they reach the grids.

A further diminishing of the resolving time is possible by a further reduction of the grid resistance. As the reduction of the grid resistance, however, causes a diminishing in size of the pulse, this procedure is only possible with simultaneous amplification of the pulses. We obtain then with a strong reduction of the duration of the pulse a sufficient size for recording it. In general one stage of amplification with a penthode is enough in order to attain the resolving time of the range of 1.10^{-6} sec. We must then take positive pulses from the cathode of the counter. Because of greater conveniency, however, in working with an earthed cathode, we added still another tube. This circuit with all its electric values is shown in fig. 2[3]. The resolving time of this circuit is 1.10^{-6} sec. Its great advantage is that the pulses coming to the grids of the Rossi tubes are equalised by the two first stages of amplification.

We are greatly indebted to the Rector of the Mining Academy of Cracow Dr W. Goetel for his interest during the execution of this work and for financial assistance, and to Professor M. Jeżewski for generous provision of the necessary facilities which enabled the work to be performed. Our thanks are also due to Mr. S. Wojtów, a skilled mechanician of the Mining Academy, for his valuable help.

References

(1) H. V. Neher and W. W. Harper, Phys. Rev. **49,** 940 (1936).

(2) A. Trost, Zs. f. Phys. **105,** 339 (1937).

(3) M. Jeżewski and M. Mięsowicz, Ann. de l'Ac. Pol. des Sc. Techn. **VII,** 41 (1939—1945).

(4) F. Rasetti, Phys. Rev. **60,** 198 (1941).

[3] During the preparation for print of the results of this investigation we have received recent American periodicals in which we have found a similar circuit in a paper by Rasetti *(4)*.

SIMPLE QUENCHING-CIRCUIT FOR G. M. COUNTERS

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(Received April 22, 1947)

E. W. Yetter *(1)* proposed a quenching circuit containing one vacuum tube connected in series with the cylinder of the counter. As the author admits, the chief disadvantages of that circuit are: (a) varying potential of the cylinder, requiring shielding and insulation if two or more counters are used, and (b) the low size of the negative output pulse. In addition a negative bias must be applied to the control grid of the quenching tube.

With the circuit shown in Fig. 1 the cylinder of the counter is on a constant potential.

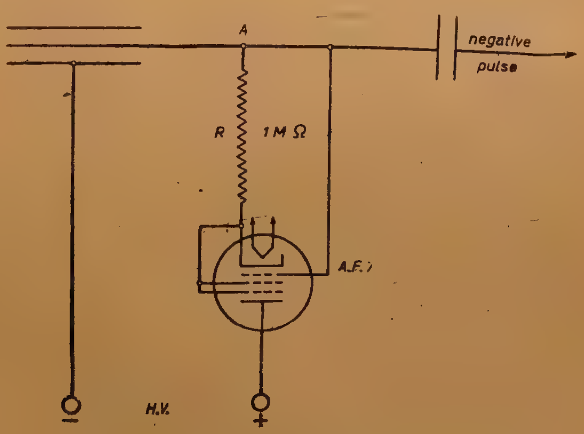

Fig. 1. Quenching Circuit Diagram.

If a sufficiently insulated filament transformer is available, each of the two high voltage supply terminals may be grounded. The principle of the operation of the circuit will be evident from the figure. A current resulting from the passage of an ionizing particle through the G.-M. counter causes a drop of potential across the vacuum tube and the resistor R. Since the potential at A, and therefore at the control grid, is then negative with respect to the cathode, the tube becomes non-conducting and the discharge stops.

Owing to the low value of the resistor R the difference of potentials between the cathode and the grid vanishes quickly causing complete recovery of the circuit. As shown by the cathode-ray oscillograph the recovery time of the system is of the order of 3×10^{-4} sec.

The advantages of the described circuit are: (1) The cylinder of the counter is on a constant potential (zero if required). (2) Similarly to the Yetters circuit, high potential is not applied across the vacuum tube and there is no constant current-drain from the high voltage supply. (3) No additional low voltage sources are needed. (4) The negative output pulses are of sufficiently high size.

The authors wish to express their grateful appreciation to Professor K. Zakrzewski for the facilities given in the course of their experiments.

Reference

(1) E. W. Yetter, Phys. Rev. **53**, 612 (1938).

AN ELECTRONIC VOLTAGE STABILIZER

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Several methods have been proposed for the stabilization of the output voltage operating G.-M. counters. A great simplicity and high constancy offers the Neher-Pickerung *(1)* circuit in which, however, bias batteries must be used.

The circuit described in this paper requires no batteries and maintains practically constant usual output voltage, even when the change of the input exceeds 50%. The circuit contains one penthode and two glow discharge tubes serving as bias batteries substitutes. By

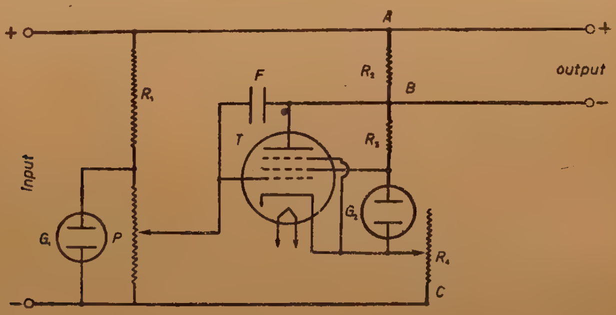

Fig. 1. $R_1 = R_2 = 3 \times 10^6$, $R_3 = 2 \times 10^6$, $R_4 = 10^5$, $P = 10^6$ ohms; $F = 0.5\,\mu F$; T = type A.F.7; G_1, G_2 = glow discharge, Philips 4357.

means of the variable resistor R_4 and the potentiometer P the cathode of the tube is brought to a positive potential with respect to the point C and the control grid — to a slightly negative potential with respect to the cathode.

As is seen from Fig. 1, the action of the penthode consists in taking upon itself the fluctuation of the input voltage. If, for instance, the input voltage increases, the potential of the cathode also increases relatively to the point C and, therefore, the difference of potentials

between the cathode and the grid increases too. The tube becomes less conductive, and this in turn produces an increase of the difference of potentials between B and C. The output voltage, taken from the points A and B, remains practically constant.

In conclusion the author wishes to acknowledge his indebtedness to Prof. K. Zakrzewski for his continued encouragement and to thank Mr. J. Janik for his technical assistance.

Reference

(1) H. V. Neher and W. H. Pickering, Rev. Sci. Inst. **10,** 53 (1939).

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